Exercise 6.1 Let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary $\partial\Omega$ of class C^1. Let $1 \leq p < \infty$. Prove that there does not exist a continuous linear operator

$$T\colon L^p(\Omega) \to L^p(\partial\Omega)$$

satisfying $Tu = u|_{\partial\Omega}$ for all $u \in C^0(\overline{\Omega})$.

Solution. We argue by contradiction. Given $k \in \mathbb{N}$, let $u_k\colon \overline{\Omega} \to \mathbb{R}$ be given by

$$u_k(x) = \begin{cases} 1 - k\,\mathrm{dist}(x, \partial\Omega), & \text{if } \mathrm{dist}(x,\partial\Omega) \leq \frac{1}{k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $u_k \in C^0(\overline{\Omega})$ and $u_k|_{\partial\Omega} \equiv 1$. Moreover, $u_k(x) \to 0$ as $k \to \infty$ for almost every $x \in \Omega$ and $|u_k| \leq 1 \in L^p(\Omega)$ for every $k \in \mathbb{N}$. Since $1 \leq p < \infty$, the dominated convergence theorem implies $\|u_k\|_{L^p(\Omega)} \to 0$ as $k \to \infty$. But $\|Tu_k\|_{L^p(\partial\Omega)} = \|1\|_{L^p(\partial\Omega)}$ does not converge to zero which contradicts continuity of T. $\square$

Exercise 6.2 Let $\Omega :=]0,1[\times]0,1[\subset \mathbb{R}^2$. Given $1 \leq p \leq \infty$, let $u \in W^{1,p}(\Omega)$.

(i) Prove that for almost every $x_2 \in]0,1[$ the function $g(x_1) = u(x_1, x_2)$ is well-defined and in $W^{1,p}(]0,1[)$ with weak derivative

$$g' = \frac{\partial u}{\partial x_1}(\cdot, x_2) \in L^p(]0,1[).$$

(ii) Suppose the weak derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ vanish almost everyhwere in Ω. Using part (i), prove that u is constant (i.e. has a constant representative).

(iii) Suppose that $v \in L^p(\Omega)$ is so that, at almost every $x \in \Omega$ the partial derivatives $\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}$ exist classically and $\frac{\partial v}{\partial x_1}(x) = \frac{\partial v}{\partial x_2}(x) = 0$. Is it still true that v is constant?

Solution. (i) In the case $1 \leq p < \infty$ we have $|u|^p \in L^1(\Omega)$ by assumption and Fubini's theorem implies that the map $x_1 \mapsto |u(x_1, x_2)|^p$ is in $L^1(]0,1[)$ for almost every $x_2 \in]0,1[$. Hence, $g := u(\cdot, x_2) \in L^p(]0,1[)$ and analogously, $f := \frac{\partial u}{\partial x_1}(\cdot, x_2) \in L^p(]0,1[)$ for almost every $x_2 \in]0,1[$.

In the case $p = \infty$ we know that $u \in W^{1,\infty}(\Omega)$ has a (globally) Lipschitz continuous representative because Ω is convex. In particular, $g = u(\cdot, x_2)$ has a Lipschitz continuous representative for almost every $x_2 \in]0,1[$. Hence, $g \in W^{1,\infty}(]0,1[)$ for almost every $x_2 \in]0,1[$.

It remains to prove that f is actually the weak derivative of g for almost all $x_2 \in \,]0,1[$. Let $\phi, \psi \in C_c^\infty(]0,1[)$ and let $\varphi(x_1,x_2) = \phi(x_1)\psi(x_2)$. Then, since $\varphi \in C_c^\infty(\Omega)$,

$$0 = \int_\Omega \frac{\partial u}{\partial x_1}\varphi + u\frac{\partial \varphi}{\partial x_1}\,dx = \int_0^1 \Big(\int_0^1 \frac{\partial u}{\partial x_1}\phi + u\phi'\,dx_1 \Big)\psi\,dx_2.$$

Since $\psi \in C_c^\infty(]0,1[)$ is arbitrary, Satz 3.4.3 (variational Lemma) applies and yields

$$\forall \phi \in C_c^\infty(]0,1[) \quad \exists G_\phi \subseteq \,]0,1[\quad \forall x_2 \in G_\phi : \quad 0 = \int_0^1 \frac{\partial u}{\partial x_1}\phi + u\phi'\,dx_1$$

and such that the Lebesgue measure of $]0,1[\, \setminus G_\phi$ vanishes for any ϕ. Let $\mathcal{P} \subset C_c^\infty(]0,1[)$ be a countable subset, which is dense in the C^1-Topology and $G = \bigcap_{\phi\in\mathcal{P}} G_\phi$. Then, since $\mathcal{P}$ is countable, the Lebesgue measure of $]0,1[\, \setminus G$ still vanishes and we obtain

$$\exists G \subseteq \,]0,1[\quad \forall \phi \in \mathcal{P} \quad \forall x_2 \in G : \quad 0 = \int_0^1 \frac{\partial u}{\partial x_1}\phi + u\phi'\,dx_1. \tag{$*$}$$

Let $\eta \in C_c^\infty(]0,1[)$ be arbitrary. By density of $\mathcal{P}$ we can choose a sequence of functions $\phi_k \in \mathcal{P}$ such that $\|\phi_k - \eta\|_{C^1} \to 0$ as $k \to \infty$ which suffices to pass to the limit in $(*)$. Hence, for all $x_2 \in G$, i.e. for almost all $x_2 \in \,]0,1[$, there holds

$$\forall \eta \in C_c^\infty(]0,1[) : \quad 0 = \int_0^1 \frac{\partial u}{\partial x_1}\eta + u\eta'\,dx_1 \quad \Rightarrow \; -\int_0^1 g\eta'\,dx_1 = \int_0^1 f\eta\,dx_1$$

which implies that f is the weak derivative of g for almost all $x_2 \in \,]0,1[$ as claimed.

(ii) If u as weak derivatives $\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_2} = 0$, by (i) and Theorem 7.3.1 for almost every x_2 there holds $u(\cdot, x_2) = C(x_2)$ almost everywhere on $]0,1[$ with $C' = 0 \in L^p(]0,1[)$ The same is true for the other variable: $u(x_1,\cdot) = \tilde{C}(x_1)$ for almost all $x_1 \in \,]0,1[$. Since $C(x_2) = \int_0^1 u(x_1,x_2)\,dx_1$ and $\tilde{C}(x_1) = \int_0^1 u(x_1,x_2)\,dx_2$ are measurable, by Fubini's theorem we have $C(x_2) = u(x_1,x_2) = \tilde{C}(x_1)$ for almost every point (x_1,x_2). Hence, for almost every fixed x_1, we have $C(x_2) = \tilde{C}(x_1)$ for almost every x_2. So C is constant almost everywhere and thus u has a constant representative.

(iii) No, consider for instance a piecewise constant function:

$$v(x_1,x_2) = \begin{cases} 1 & \text{if } x_1 > 0, \\ -1 & \text{if } x_1 \le 0. \end{cases}$$

Then v is classically differentiable away from the line $\{x_1 = 0\}$ with $dv \equiv 0$, but it is not constant. $\square$

Exercise 6.3 Let

$$B_1(0) = \Big\{z : |z| < 1\Big\} = \Big\{re^{i\theta} : r \in (0,1), \theta \in [0, 2\pi)\Big\} \subset \mathbb{R}^2 = \mathbb{C}$$

be the unit disk in the complex plane and let

$$S^1 = \Big\{z : |z| = 1\Big\} = \Big\{e^{i\theta} : \theta \in [0, 2\pi)\Big\}$$

be its boundary. We may identify real or complex-valued functions $\varphi : S^1 \to \mathbb{C}$ and 2π-periodic functions simply by identifying $\varphi(\theta) \simeq \varphi(e^{i\theta})$ for $\theta \in \mathbb{R}$.

Here are three reminiscences from basic complex analysis:

Fact 1. A holomorphic function $f : B_1(0) \to \mathbb{C}$ can be written as a complex power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in B_1(0),$$

uniformly convergent over compact subset over $B_1(0)$. Viceversa, any power series with the above property defines uniquely a holomorphic function over $B_1(0)$.

Fact 2. The real and complex part of f are harmonic functions.

Fact 3. Any harmonic function $u : B_1(0) \to \mathbb{R}$ possesses a harmonic conjugate, i.e. a harmonic function $v : B_1(0) \to \mathbb{R}$ so that $f = u + iv$ is holomorphic. Such v is unique up to an additive constant.

Prove the following.

(i) Let $u_0 : S^1 \to \mathbb{R}$ be a smooth function. Extend u as a harmonic function to the disc $B_1(0)$ by solving

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0), \\ \quad u = u_0 & \text{on } S^1. \end{cases}$$

Express u in terms of the coefficients of the Fourier series of u_0 (see Exercise 5.1), wiriting it in a form of a series, using polar polar coordinates (r, θ). Then, express this in a series in complex coordinates $z, \bar{z}$. (recall: $(x^1, x^2) \simeq x^1 + ix^2 = z = re^{i\theta}$, $\bar{z} = x^1 - x^2$).

(ii) Recall that using Green's function for $B_1(0)$, we also have the integral representation of u as

$$u(re^{i\theta}) = \int_0^{2\pi} P(r, \theta - \alpha) u_0(\alpha) d\alpha,$$

where

$$P(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}$$

is *Poisson's kernel.* Verify that this is consistent with the formula found in (i) deducing one from another.

(iii) With the help of (i) find an expression in complex coordinates for the unique harmonic conjugate of u vanishing at 0 in terms of the Fourier coefficients of u_0.

(iv) Now let v be the harmonic conjugate of u found in (iv). Argue as in (ii) to obtain an integral representation for v in terms of an integral over S^1 involving u_0:

$$v(r\mathrm{e}^{i\theta}) = \int_0^{2\pi} Q(r,\alpha)u_0(\alpha)d\alpha,$$

finding explicitly the function Q, known as the *conjugate Poisson kernel.* What happens to this integral when you consider, for any fixed $\theta_0 \in [0,2\pi)$,

$$\lim_{\substack{r\to 1^- \\ \theta\to\theta_0}} v(r\mathrm{e}^{i\theta}) \;...?$$

Solution. (i) Writing u_0 as Fourier series

$$u_0(\mathrm{e}^{i\theta}) = u_0(\theta) = \sum_{n\in\mathbb{Z}} \widehat{u_0}(n)\mathrm{e}^{in\theta},$$

since for every $n \geq 0$ the functions

$$r^n\mathrm{e}^{in\theta} = z^n, \quad r^n\mathrm{e}^{-in\theta} = \bar{z}^n,$$

are, respectively, holomorphic and antiholomorphic, and thus harmonic, and harmonic extension of u_0 is unique, we necessarily have

$$u(r\mathrm{e}^{i\theta}) = \sum_{n\in\mathbb{Z}} \widehat{u_0}(n)r^{|n|}\mathrm{e}^{in\theta},$$

and from this we see that the complex representation in term of z and $\bar{z}$ is

$$u(z) = \sum_{n=0}^{\infty} \widehat{u_0}(n)z^n + \sum_{n=1}^{\infty} \widehat{u_0}(-n)\bar{z}^n$$

All the series are uniformly convergent for $r \leq 1$ since so the Fourier series of u_0.

(ii) Recall that we have

$$\widehat{u_0}(n) = \frac{1}{2\pi}\int_0^{2\pi} u_0(\alpha)\mathrm{e}^{-in\alpha}d\alpha,$$

we may rewrite, for any $r < 1$, with a change of variables,

$$\begin{aligned} u(r\mathrm{e}^{i\theta}) &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_0^{2\pi} u_0(\alpha)\mathrm{e}^{in(\theta-\alpha)}r^{|n|}d\alpha \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_0^{2\pi} u_0(\theta-\alpha)\mathrm{e}^{in\alpha}r^{|n|}d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi} u_0(\theta-\alpha)\sum_{n\in\mathbb{Z}}\mathrm{e}^{in\alpha}r^{|n|}d\alpha \end{aligned}$$

where we could interchange the summation and the integration operation because of uniform convergence. We can easily work out explicitly the series by working in complex notation:

$$\sum_{n\in\mathbb{Z}}\mathrm{e}^{in\alpha}r^{|n|} = \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty}\bar{z}^n = \frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{1-|z|^2}{|1-z|^2},$$

and since $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, we have

$$\frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{1+r^2-2r\cos\alpha}.$$

We deduce the validity of the integral formula

$$\begin{aligned} u(r\mathrm{e}^{i\theta}) &= \frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{u_0(\theta-\alpha)}{1+r^2-2r\cos\theta}d\alpha \\ &= \frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{u_0(\alpha)}{1+r^2-2r\cos(\theta-\alpha)}d\alpha. \end{aligned}$$

One verifies that this is exactly the representation of u using the Green function.

(iii) We need to find v so that $u + iv$ is a complex power series. Since v has to be harmonic, it will have, similarly as for u, the form

$$v(z) = \sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} c_{-n}\bar{z}^n,$$

for appropriate coefficients c_n's. Since $u+iv$ must have vanishing antiholomorphic part, we deduce

$$ic_{-n} = -\widehat{u_0}(-n) \quad \text{for } n \geq 1.$$

On the other hand, v must be real valued, hence $\overline{v} = v$ and this implies

$$c_n = \overline{c_{-n}} = \overline{i\widehat{u_0}(-n)} = -i\overline{\widehat{u_0}(-n)} = -i\widehat{u_0}(n) \quad \text{for } n \geq 1.$$

Finally the requirement that $v(0) = 0$ is equivalent to $c_0 = 0$. We conclude that

$$v(z) = -i\bigg(\sum_{n=1}^{\infty} \widehat{u_0}(n) z^n - \sum_{n=1}^{\infty} \widehat{u_0}(-n) \bar{z}^n \bigg).$$

(iv) We argue similarly as in (ii). For any fixed $r < 1$ we have, in polar coordinates (below it is, by convention, $\operatorname{sign}(0) = 0$):

$$\begin{aligned} v(r\mathrm{e}^{i\theta}) &= -i\bigg(\sum_{n=1}^{\infty} \widehat{u_0}(n) r^n e^{in\theta} - \sum_{n=1}^{\infty} \widehat{u_0}(-n) r^n e^{-in\theta} \bigg) \\ &= -i\bigg(\sum_{n\in\mathbb{Z}} \widehat{u_0}(n) \operatorname{sign}(n) r^{|n|} e^{in\theta} \bigg) \\ &= -i\bigg(\sum_{n\in\mathbb{Z}} \frac{1}{2\pi} \int_0^{2\pi} u_0(\alpha) \mathrm{e}^{in(\theta-\alpha)} \operatorname{sign}(n) r^{|n|} d\alpha \bigg) \\ &= -i\bigg(\sum_{n\in\mathbb{Z}} \frac{1}{2\pi} \int_0^{2\pi} u_0(\theta-\alpha) \mathrm{e}^{in\alpha} \operatorname{sign}(n) r^{|n|} d\alpha \bigg) \\ &= -i\bigg(\frac{1}{2\pi} \int_0^{2\pi} u_0(\theta-\alpha) \sum_{n\in\mathbb{Z}} \mathrm{e}^{in\alpha} \operatorname{sign}(n) r^{|n|} d\alpha \bigg); \end{aligned}$$

now it is

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{in\alpha} \operatorname{sign}(n) r^{|n|} = \sum_{n=1}^{\infty} z^n - \sum_{n=1}^{\infty} \bar{z}^n = \frac{z}{1-z} - \frac{\bar{z}}{1-\bar{z}} = \frac{z-\bar{z}}{1-z-\bar{z}+|z|^2},$$

which reads in polar coordinates as

$$\frac{z-\bar{z}}{1-z-\bar{z}+|z|^2} = \frac{2ir\sin\alpha}{1-2r\cos\alpha+r^2}.$$

Thus, it is

$$\begin{aligned} v(r\mathrm{e}^{i\theta}) &= \frac{r}{\pi} \int_0^{2\pi} \frac{\sin\alpha}{1-2r\cos\alpha+r^2} u_0(\theta-\alpha) d\alpha \\ &= \frac{r}{\pi} \int_0^{2\pi} \frac{\sin(\theta-\alpha)}{1-2r\cos(\theta-\alpha)+r^2} u_0(\alpha) d\alpha, \end{aligned}$$

and the cojugate Poisson kernel is then

$$Q(r,\theta) = \frac{r}{\pi} \frac{r\sin\alpha}{1-2r\cos\alpha+r^2}$$

As we let $r \to 1^-$, then

$$Q(r, \theta) \to \frac{1}{\pi} \frac{\sin\theta}{2(1 - \cos\theta)},$$

and this expression is singular as $\theta \to 0$, since

$$\frac{\sin\theta}{2(1-\cos\theta)} \underset{\theta \to 0}{\sim} \frac{1}{\theta},$$

and $\frac{1}{\theta}$ is not even locally integrable around zero. Thus, the integral defining the boundary value of v is "singular", i.e. not absolutely convergent even for smooth u_0. $\square$

Hints to Exercises.

6.1 Find a sequence of functions $u_k \in C^0(\overline{\Omega})$ satisfying $u_k|_{\partial\Omega} \equiv 1$ and $\|u_k\|_{L^p(\Omega)} \xrightarrow{k\to\infty} 0$.

6.2 For (i): when $1 \le p < \infty$ apply Fubini's theorem and for $p = \infty$, argue with Lipschitz continuity (Korollar 8.3.1).

For (ii): Apply part (i) and use Lemma 7.3.1.

6.3 For (i): note that $r^n \mathrm{e}^{in\theta} = z^n$ and $r^n \mathrm{e}^{-in\theta} = \bar{z}^n$ are (complex) harmonic functions, thus since the harmonic extension is unique...

For (ii): recall the usual formulas for (real or complex) geometric series: if $|z| < 1$, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, $\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$, ecc.

For (iii): Find v in the form $v(z) = \sum c_n z^n + \sum c_{-n} \bar{z}^n$ so that the sum $u + iv$ has no $\bar{z}$-terms.