Exercise 7.1 Let $1 \leq p \leq \infty$. Consider the open set

$$\Omega = (-1,1) \times (-1,1) \setminus \Big([0,1) \times \{0\}\Big) \subset \mathbb{R}^2.$$

Prove that there is no extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$.

Solution. Let $u\colon \Omega \to \mathbb{R}$ be given by

$$u(x_1, x_2) := \begin{cases} x_1 & \text{if } x_1 > 0 \text{ and } x_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

be the function as in in Exercise 5.2. We showed there that $u \in W^{1,\infty}(\Omega)$. Since Ω is bounded, $u \in W^{1,p}(\Omega)$ for any $1 \leq p \leq \infty$. Suppose, there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ such that $(Eu)|_\Omega = u$ almost everywhere in Ω. Let $Q := (-1,1) \times (-1,1)$ and $v := (Eu)|_Q$. Then $Eu \in W^{1,p}(\mathbb{R}^n)$ implies $v \in W^{1,p}(Q)$. Consequently (as shown in Exercise 6.2) $(x_2 \mapsto v(x_1, x_2)) \in W^{1,p}((-1,1))$ for almost every $x_1 \in (-1,1)$. Moreover, since $[0,1) \times \{0\}$ has measure zero, $v(x_1, x_2) = u(x_1, x_2)$ for almost every $(x_1, x_2) \in Q$.

Hence, there exists some fixed $x_1 \in (\frac{1}{2}, 1)$ such that $(g\colon x_2 \mapsto v(x_1, x_2)) \in W^{1,p}((-1,1))$ and such that $g(x_2) = u(x_1, x_2)$ for almost every $x_2 \in (-1,1)$. By Sobolev's embedding in dimension one, g and hence $x_2 \mapsto u(x_1, x_2)$ has a representative in $C^0((-1,1))$. However, since we chose $x_1 > \frac{1}{2}$, this leads to a contradiction since

$$x_2 \mapsto u(x_1, x_2) = \begin{cases} x_1 & \text{for } x_2 > 0, \\ 0 & \text{for } x_2 < 0. \end{cases}$$

is not continuous. □

Exercise 7.2 In this exercise we want to prove that, for every bounded, C^1 domain $\Omega \subset \mathbb{R}^n$ and every $1 \leq p < \infty$, $W_0^{1,p}(\Omega)$ consists *exactly* of those functions in $W^{1,p}(\Omega)$ with vanishing trace, similarly to Remark 7.5.1 in the 1-dimensional case or Corollary 8.4.3 for the case $p = 2$.

Let $u \in W^{1,p}(\Omega)$.

(i) Prove that for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ and every $i = 1, \ldots, n$ there holds

$$\int_\Omega \partial_i u \, \varphi \, dx = -\int_\Omega u \, \partial_i \varphi \, dx + \int_{\partial\Omega} u|_{\partial\Omega} \, \varphi \, \nu^i \, d\sigma,$$

where $\nu = (\nu^1, \ldots, \nu^n)$ denotes the outer unit normal of $\partial\Omega$ and $u|_{\partial\Omega} \in L^p(\partial\Omega)$ denotes the trace of u.

(ii) Consider the extension of U by zero to $\mathbb{R}^n$:

$$U(x) = \begin{cases} u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Prove that, if the weak derivative of U exist, they are necessarily given by

$$\partial_i U(x) = \begin{cases} \partial_i u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega \end{cases} \tag{$*$}$$

for $i = 1, \ldots, n$.

Prove then that $u|_{\partial\Omega} = 0$ if and only if U is in $W^{1,p}(\mathbb{R}^n)$.

(iii) Prove that, for every $v \in W^{1,p}(\mathbb{R}^n)$ so that $v|_{\mathbb{R}^n \setminus \Omega} = 0$ then $v|_\Omega \in W_0^{1,p}(\Omega)$ and conclude.

Solution. (i) For $u \in C^\infty(\mathbb{R}^n)$ it is the classical integration by parts formula. For a general $u \in W^{1,p}(\Omega)$, since Ω is a regular domain we can argue by approximation: let $(u_j)_j \subset C_c^\infty(\mathbb{R}^n)$ be a sequence of smooth functions so that $u_j \to u$ in $W^{1,p}(\Omega)$ as $j \to \infty$. Then we have

$$\int_\Omega \partial_i\, u_j \varphi \, dx = -\int_\Omega \partial_i u_j \varphi \, dx + \int_{\partial\Omega} u_j|_{\partial\Omega} \, \varphi \, \nu^i \, d\sigma.$$

We may now pass to the limit in this expression: this follows from definition of $W^{1,p}$-convergence for the integrals over Ω, while for the boundary integral we have that $u_j|_{\partial\Omega}$ converges to $u|_{\partial\Omega}$ in $L^p(\Omega)$ since the trace operator is continuous from $W^{1,p}(\Omega)$ to $L^p(\Omega)$.

(ii) If the weak derivatives of U exist, then they must be the functions given $(*)$ above, since $\partial\Omega$ has zero measure.

Consequently, on the one hand if u has vanishing trace, then the formula in (i) is telling us exactly that the weak derivatives of U exist, and thus also that $U \in W^{1,p}(\mathbb{R}^n)$.

Vice versa, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, by (i) we have

$$\begin{aligned}\int_{\mathbb{R}^n} U \, \partial_i \varphi \, dx &= \int_\Omega u \, \partial_i \varphi \, dx \\ &= -\int_\Omega \partial_i u \, \varphi \, dx + \int_{\partial\Omega} u|_{\partial\Omega} \, \varphi \, \nu^i d\sigma \\ &= -\int_{\mathbb{R}^n} \partial_i U \, \psi \, dx + \int_{\partial\Omega} u|_{\partial\Omega} \, \varphi \, \nu^i d\sigma.\end{aligned}$$

Thus, for U to have weak derivatives it is necessariy that, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, $i \in \{1, \dots, n\}$ there holds

$$\int_{\partial\Omega} u|_{\partial\Omega}\, \varphi\, \nu^i d\sigma = 0,$$

and so $u|_{\partial\Omega}$ must vanish.

(iii) *Step 1.* The problem can be reduced to the following model case. Let

$$\begin{aligned} Q &= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x'| < 1 \text{ and } |x_n| < 1\}, \\ Q_+ &= \{x = (x', x_n) \in Q \mid x_n > 0\}, \\ Q_0 &= \{x = (x', x_n) \in Q \mid x_n = 0\}. \end{aligned}$$

Let $u \in W^{1,p}(Q)$ satisfy $u = 0$ in $Q \setminus Q_+$. Then we claim $\alpha u \in W_0^{1,p}(Q_+)$ for any $\alpha \in C_c^1(Q)$. Note that since α is compactly supported in Q, (αu) extends to a function in $W^{1,p}(\mathbb{R}^n)$ which allows mollification. Let $0 \le \rho \in C_c^\infty(B_1(0))$ satisfy

$$\operatorname{supp}(\rho) \subset \{(x', x_n) \in B_1(0) \mid \tfrac{1}{2} < x_n < 1\}, \qquad \int_{B_1(0)} \rho\, dx = 1$$

and let $\rho_m(x) := m^n \rho(mx)$ for $m \in \mathbb{N}$. Then, $\|\rho_m * (\alpha u) - (\alpha u)\|_{W^{1,p}} \to 0$ as $m \to \infty$. Moreover, if $x = (x', x_n) \in Q_+$ with $x_n < \frac{1}{4m}$ then $(\alpha u)(x - y) = 0$ whenever $y_n > \frac{1}{2m}$ because u vanishes outside Q_+. Hence, by choice of $\operatorname{supp}(\rho_m)$,

$$\Big(\rho_m * (\alpha u)\Big)(x) = \int_{\mathbb{R}^n} \rho_m(y)\, (\alpha u)(x - y)\, dy = 0 \quad \text{ if } x_n < \tfrac{1}{4m}$$

which implies $\rho_m * (\alpha u) \in C_c^\infty(Q_+)$ and therefore $\alpha u \in W_0^{1,p}(Q_+)$.

Step 2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary of class C^1. Since $\partial\Omega$ is compact and regular, there exist finitely many open sets $U_1, \dots, U_N \subset \mathbb{R}^n$ and diffeomorphisms $H_k \colon Q \to U_k$ such that for every $k \in \{1, \dots, N\}$

$$H_k(Q_+) = U_k \cap \Omega, \qquad H_k(Q_0) = U_k \cap \partial\Omega, \qquad \partial\Omega \subset \bigcup_{k=1}^N U_k.$$

Furthermore, there exists an open set $U_0 \subset \mathbb{R}^n$ such that $\overline{U_0} \subset \Omega$ and $\Omega \subset \bigcup_{k=0}^N U_k$. Let $(\varphi_k)_{k\in\{0,\dots,N\}}$ be a corresponding partition of unity, i. e. a collection of smooth functions such that for every $k \in \{0, \dots, N\}$

$$0 \le \varphi_k \le 1, \qquad \operatorname{supp}(\varphi_k) \subset U_k, \qquad \sum_{k=0}^N \varphi_k|_\Omega = 1.$$

Let $v \in W^{1,p}(\mathbb{R}^n)$ satisfy $v(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. By Satz 8.3.3, $v \circ H_k \in W^{1,p}(Q)$ for $k \in \{1, \dots, N\}$ and it satisfies $v \circ H_k = 0$ in $Q \setminus Q_+$. By Step 1, choosing $\alpha = \varphi_k \circ H_k$, we have $(\varphi_k v) \circ H_k \in W_0^{1,p}(Q_+)$ Let $w_k^{(m)} \in C_c^\infty(Q_+)$ be such that $\|w_k^{(m)} - (\varphi_k v) \circ H_k\|_{W^{1,p}(Q_+)} \to 0$ as $m \to \infty$. Moreover, since $\operatorname{supp}(\varphi_0) \subset U_0 \subset \Omega$, we can approximate $\varphi_0 v$ by $v_0^{(m)} \in C_c^\infty(\Omega)$ directly using mollification. Then, we have

$$w^{(m)} := v_0^{(m)} + \sum_{k=1}^{N} (w_k^{(m)} \circ H_k^{-1}) \in C_c^\infty(\Omega)$$

and since $v = \sum_{k=0}^{N} \varphi_k v$ in Ω by partition of unity,

$$\begin{aligned}
&\|w^{(m)} - v\|_{W^{1,p}(\Omega)} \\
&\leq \|v_0^{(m)} - \varphi_0 v\|_{W^{1,p}(\Omega)} + \sum_{k=1}^{N} \left\| w_k^{(m)} \circ H_k^{-1} - \varphi_k v \right\|_{W^{1,p}(\Omega)} \\
&\leq \|v_0^{(m)} - \varphi_0 v\|_{W^{1,p}(\Omega)} + \sum_{k=1}^{N} C \left\| w_k^{(m)} - (\varphi_k v) \circ H_k \right\|_{W^{1,p}(Q_+)} \xrightarrow{m \to \infty} 0
\end{aligned}$$

which concludes the proof of $v|_\Omega \in W_0^{1,p}(\Omega)$. □

Exercise 7.3 Show that the assumption that Ω is of class C^1 cannot be dropped in the characterization of $W_0^{1,p}(\Omega)$ given in Exercise 7.2: find a bounded, connected, open set $\Omega \subset \mathbb{R}^2$ and $w \in H^1(\mathbb{R}^2)$ satisfying $w(x) = 0$ for almost every $x \in \mathbb{R}^2 \setminus \Omega$ such that $w|_\Omega \notin H_0^1(\Omega)$.

Solution. Let $\Omega = (-1,1)^2 \setminus ([0,1) \times \{0\})$ and let $u \in C^\infty(\mathbb{R}^n)$ satisfy $u(x) = 1$ if $|x| < \frac{1}{2}$ and $u(x) = 0$ if $|x| > \frac{3}{4}$. Then $u \in H^1(\Omega)$ and $u(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. Suppose by contradiction that there exists a sequence of functions $u_m \in C_c^\infty(\Omega)$ such that $\|u_m - u\|_{H^1(\Omega)} \to 0$ as $m \to \infty$. Let $Q := (0,1)^2$ and $Q_0 = (0,1) \times \{0\}$. By Lemma 8.4.2 the trace operator $T \colon H^1(Q) \to L^2(Q_0)$ mapping $T \colon u \mapsto u|_{Q_0}$ is linear and continuous. In particular,

$$\|Tu_m - Tu\|_{L^2(Q_0)} \leq C \|u_m - u\|_{H^1(Q)} \xrightarrow{m \to \infty} 0.$$

Since $Q_0 \subset \partial\Omega$ implies $Tu_m = u_m|_{Q_0} = 0$, we obtain $u|_{Q_0} = 0$ in $L^2(Q_0)$. This however contradicts the fact that $u(x) = 1$ for $|x| < \frac{1}{2}$. □

Exercise 7.4 (Hardy's inequalities)

(i) Let $1 < p < \infty$, let $f \in L^p((0,\infty))$ and define

$$g(x) = \frac{1}{x}\int_0^x f(y)dy, \quad \text{for } x > 0.$$

Prove that $g \in L^p((0,\infty))$ with

$$\|g\|_{L^p((0,\infty))} \leq C\|f\|_{L^p((0,\infty))},$$

for some constant $C > 0$ depending only on p.

(ii) Let $n \geq 2$, $1 < p < n$, $\Omega \subseteq \mathbb{R}^n$ be an open subset and let $u \in W_0^{1,p}(\Omega)$. Then the function $x \mapsto \frac{u(x)}{|x|}$ is in $L^p(\Omega)$ with

$$\left\|\frac{u}{|\cdot|}\right\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

for a constant $C > 0$ depending only on n and p.

Solution. (i) With a change of variable $y' = \frac{y}{x}$ we may write

$$g(x) = \int_0^1 f(xy)dy.$$

Consequently, using Minkowski's inequality for integrals and the change of variable $z = xy$, we have

$$\begin{aligned}
\|g\|_{L^p((0,\infty))} &= \left(\int_0^\infty |g(x)|^p dx\right)^{1/p} \\
&= \left(\int_0^\infty \left|\int_0^1 f(xy)dy\right|^p dx\right)^{1/p} \\
&\leq \int_0^1 \left(\int_0^\infty |f(xy)|^p dx\right)^{1/p} dy \\
&= \int_0^1 \frac{1}{y^{1/p}}\left(\int_0^\infty |f(z)|^p dz\right)^{1/p} dy = \frac{p}{p-1}\|f\|_{L^p((0,\infty))}.
\end{aligned}$$

(ii) Let us first prove the inequality for $u \in C_c^\infty(\mathbb{R}^n)$. Using polar coordinates, each $x \in \mathbb{R}^n$, $x \neq 0$ can be uniquely written as $x = |x|\theta_x$, where $\theta_x = \frac{x}{|x|} \in S^{n-1}$.

By the fundamental theorem of calculus, since u vanishes at infinity we may represent u as

$$u(x) = -\int_{|x|}^{\infty} \frac{\partial u}{\partial r}(r\theta_x)dr, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where then $\frac{\partial}{\partial r}$ is the radial derivative. Thus, with a change of variable $\rho = \frac{r}{|x|}$, Minkowski's inequality for integrals and again a change of variables $z = \rho x$, we see that

$$\begin{aligned}
\left\| \frac{u}{|\cdot|} \right\|_{L^p(\Omega)} &= \left(\int_{\mathbb{R}^n} \left| \frac{1}{|x|} \int_{|x|}^{\infty} \frac{\partial u}{\partial r}(r\theta_x)dr \right|^p dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n} \left| \int_1^{\infty} \frac{\partial u}{\partial r}(\rho x)d\rho \right|^p dx \right)^{1/p} \\
&\le \int_1^{\infty} \left(\int_{\mathbb{R}^n} \left| \frac{\partial u}{\partial r}(\rho x) \right|^p dx \right)^{1/p} d\rho \\
&= \int_1^{\infty} \frac{1}{\rho^{n/p}} \left\| \frac{\partial u}{\partial r} \right\|_{L^p(\mathbb{R}^n)} d\rho = \frac{p}{n-p} \left\| \frac{\partial u}{\partial r} \right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

It now suffices to recall that gradient in polar coordinates is expressed as

$$|\nabla u| = \left| \frac{\partial u}{\partial r} \right| + \frac{1}{r} \left| \frac{\partial u}{\partial \theta} \right|,$$

hence $\|\partial_r u\|_{L^p(\mathbb{R}^n)} \le \|\nabla u\|_{L^p(\mathbb{R}^n)}$ and this implies in particular the desired inequality in the case $u \in C_c^\infty(\mathbb{R}^n)$.

Now if $u \in W_0^{1,p}(\Omega)$, we may argue by approximation, noting that if $(u_j)_j \subset C_c^\infty(\mathbb{R}^n)$ is a sequence approximating u in $W^{1,p}(\Omega)$, then $x \mapsto \frac{u_j(x)}{|x|}$ converges a.e. to $x \mapsto \frac{u(x)}{|x|}$ and thus, since the sequence is also convergent in L^p, so is its pointwise limit. $\square$

Hints to Exercises.

7.1 Recall Exercise 5.2.

7.2 For (iii), deal first with the basic case on cylinders.

7.3 Compare with Exercises 5.2 and 7.1.

7.4 Minkowski inequality for integrals: $\| \int f(x, \cdot) dx \|_{L^p} \leq \int \|f(x, \cdot)\|_{L^p} dx$ will be useful.

For (ii), argue first for $u \in C_c^\infty(\mathbb{R}^n)$ and write u as integral of its radial derivative $u(x) = -\int_{|x|}^\infty \partial_r u(r\theta_x) dr$.