Exercise 8.1 Let $n \in \mathbb{N}$.

(i) Prove that the embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$$

is *never* compact for any $1 \leq p \leq \infty$.

(ii) Let $1 < p \leq n$ and $\Omega \subset \mathbb{R}^n$ a domain of finite Lebesgue measure, possibly unbounded. Prove that the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

is compact.

(iii) Is the statement of part (ii) still true if the space $W_0^{1,p}(\Omega)$ is replaced by $W^{1,p}(\Omega)$?

Solution. (i) Let $u \in C_c^\infty(\mathbb{R}^n)$ satisfy $\|u\|_{W^{1,p}(\mathbb{R}^n)} = 1$. For any $k \in \mathbb{N}$, let

$$u_k(x) = u(x + ke_1),$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Then $\|u_k\|_{W^{1,p}(\mathbb{R}^n)} = 1$ for every $k \in \mathbb{N}$. Assume by contradiction that the embedding in question is compact. Up to extracting a subsequence $(u_k)_{k\in\mathbb{N}}$ is then convergent $L^p(\mathbb{R}^n)$ to some function v, and hence, again up to extracting a subsequence, it also converges pointwise a.e. However for every $x \in \mathbb{R}^n$ there always exists $k_0 \in \mathbb{N}$ so that $u_k(x) = 0$ for $k \geq k_0$ and thus it must be $v = 0$ a.e. and this is absurd since

$$1 = \|u\|_{L^p(\mathbb{R}^n)} = \|u_k\|_{L^p(\mathbb{R}^n)} \xrightarrow{k\to\infty} \|v\|_{L^p(\mathbb{R}^n)} = 0.$$

$\square$

(ii) Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,p}(\Omega)$ satisfying $\|u_k\|_{W^{1,p}(\Omega)} \leq C$ for every $k \in \mathbb{N}$. In particular, $u_k \in W_0^{1,p}(\Omega)$ can be extended *by zero* to a function $\overline{u}_k \in W^{1,p}(\mathbb{R}^n)$. Thus, $\|\overline{u}_k\|_{W^{1,p}(\mathbb{R}^n)} \leq C$ for every $k \in \mathbb{N}$. Since $1 < p < \infty$, the space $W^{1,p}(\mathbb{R}^n)$ is reflexive and hence up to passing to a subsequence, $(\overline{u}_k)_{k\mathbb{N}}$ converges weakly to some $v \in W^{1,p}(\mathbb{R}^n)$.

Now for any $R > 0$ the embedding $W^{1,p}(B_R) \hookrightarrow L^p(B_R)$ is compact, hence $(\overline{u}_k|_{B_R})_{k\in\mathbb{N}}$ has a converging subsequence $L^p(B_R)$. Restricting to nested subsequences for each $R \in \mathbb{N}$ and choosing a diagonal sequence, up to passing to a further subsequence we have that $(\overline{u}_k|_{B_R})_{k\in\mathbb{N}}$ converges in $L^p(B_R)$ for any $R \in \mathbb{N}$. The limit must coincide with $v|_{B_R}$ by uniqueness of weak limits: both, weak convergence in $W^{1,p}$ and norm-convergence in L^p, imply weak convergence in L^p.

Let us prove that then $(u_k)_{k\in\mathbb{N}}$ converges to v in $L^p(\Omega)$.

Note first the following:

If $p<n$, then Sobolev's embedding $W^{1,p}(\mathbb{R}^n)\hookrightarrow L^{p^*}(\mathbb{R}^n)$ implies

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_R}|\overline{u}_k|^p\,dx &= \int_{\Omega\setminus B_R}|u_k|^p\,dx\\ &\le \Big(\int_{\mathbb{R}^n}|\overline{u}_k|^{p^*}\,dx\Big)^{\frac{p}{p^*}}|\Omega\setminus B_R|^{\frac{p}{n}} && \text{(Hölder)}\\ &\le C\,\|\nabla\overline{u}_k\|^p_{L^p(\mathbb{R}^n)}|\Omega\setminus B_R|^{\frac{p}{n}} && \text{(Sobolev)}\\ &\le C|\Omega\setminus B_R|^{\frac{p}{n}}.\end{aligned}$$

If $p=n$, then $W^{1,n}(\mathbb{R}^n)\hookrightarrow L^q(\mathbb{R}^n)$ for any $n\le q<\infty$, in particular for $q=2n$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_R}|\overline{u}_k|^n\,dx &= \int_{\Omega\setminus B_R}|u_k|^n\,dx\\ &\le \Big(\int_{\mathbb{R}^n}|\overline{u}_k|^{2n}\,dx\Big)^{\frac{1}{2}}|\Omega\setminus B_R|^{\frac{1}{2}} && \text{(Hölder)}\\ &\le C\,\|\overline{u}_k\|^n_{W^{1,n}(\mathbb{R}^n)}|\Omega\setminus B_R|^{\frac{1}{2}} && \text{(Sobolev)}\\ &\le C|\Omega\setminus B_R|^{\frac{1}{2}}.\end{aligned}$$

The same estimates also hold for $v\in W^{1,p}(\mathbb{R}^n)$ in place of $\overline{u}_k$. Let $\varepsilon>0$ be arbitrary. Since $|\Omega|<\infty$, the estimates above imply that there exists some $R_\varepsilon\in\mathbb{N}$ such that

$$\forall k\in\mathbb{N}:\quad \|\overline{u}_k\|^p_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})}<\varepsilon,\qquad\qquad \|v\|^p_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})}<\varepsilon.$$

By also choosing $N_\varepsilon\in\mathbb{N}$ such that $\|\overline{u}_k-v\|^p_{L^p(B_{R_\varepsilon})}<\varepsilon$ for every $k\ge N_\varepsilon$, we get

$$\begin{aligned}\|u_k-v\|^p_{L^p(\Omega)} &\le \|\overline{u}_k-v\|^p_{L^p(\mathbb{R}^n)} = \|\overline{u}_k-v\|^p_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})}+\|\overline{u}_k-v\|^p_{L^p(B_{R_\varepsilon})}\\ &\le \Big(\|\overline{u}_k\|_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})}+\|v\|_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})}\Big)^p+\|\overline{u}_k-v\|^p_{L^p(B_{R_\varepsilon})}\\ &< (2^p+1)\varepsilon.\end{aligned}$$

Hence, $(u_k)_{k\in\mathbb{N}}$ converges to v in $L^p(\Omega)$ and so the embedding $W_0^{1,p}(\Omega)\hookrightarrow L^p(\Omega)$ is indeed compact.

(iii) No, in general. Consider for for $n\ge 2$ the domain $\Omega\subset\mathbb{R}^n$ given by

$$\Omega := \bigcup_{k=2}^{\infty}\Omega_k \quad\text{with}\quad \Omega_k = B_{\frac{1}{k}}(ke_1),$$

where where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Then Ω has finite measure:

$$|\Omega| = |B_1| \sum_{k=2}^{\infty} k^{-n} < \infty,$$

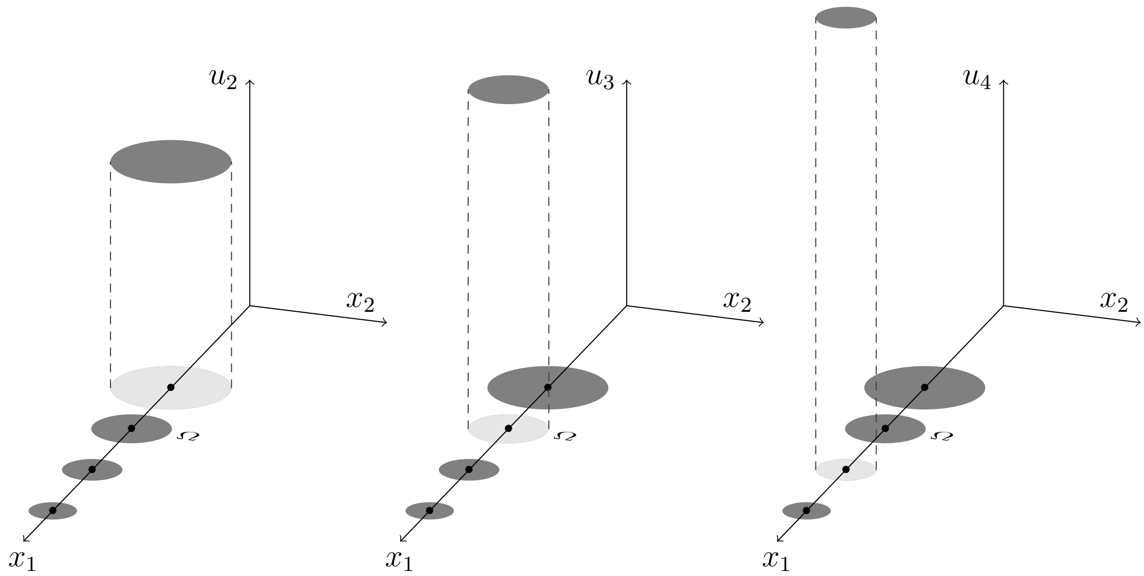
and we can consider the sequence

$$u_k = \begin{cases} k^{\frac{n}{p}} & \text{in } \Omega_k, \\ 0 & \text{else.} \end{cases}$$

Clearly $u_k \in W^{1,p}(\Omega)$ and

$$\|u_k\|^p_{W^{1,p}(\Omega)} = \|u_k\|^p_{L^p(\Omega)} = |B_{\frac{1}{k}}| k^n = |B_1| \qquad \forall k \geq 2,$$

moreover it converges to 0 a.e. in Ω. Thus, agrguing as in (i), it follows that it does not admit any converging subsequence in $L^p(\Omega)$.



Exercise 8.2 (Ehrling's Lemma)

(i) Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces with continuous embeddings

$$X \hookrightarrow Y \hookrightarrow Z$$

and so that the embedding

$$X \hookrightarrow Y$$

is compact. Prove that then for every $\varepsilon > 0$ there exist $C_\varepsilon > 0$ so that

$$\|x\|_Y \leq \varepsilon \|x\|_X + C_\varepsilon \|x\|_Z \quad \forall x \in X.$$

(ii) Let now $\Omega \subset \mathbb{R}^n$ be a bounded, regular domain, let $1 \leq p \leq \infty$ and let $u \in W^{2,p}(\Omega)$, $f \in L^p(\Omega)$. Suppose you know that the following inequality holds:

$$\|u\|_{W^{2,p}(\Omega)} \leq C\, \Big(\|f\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)}\Big), \tag{$*$}$$

where $C > 0$ does not depend on u or f. Use (i) to deduce that in fact there holds

$$\|u\|_{W^{2,p}(\Omega)} \leq C'\, \Big(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\Big),$$

for some constant $C' = C'(C, n, \Omega) > 0$.

Solution. (i) Suppose by contradiction that there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X and some $\varepsilon_0 > 0$ such that for every $k \in \mathbb{N}$ there holds

$$1 = \|x_k\|_Y \geq \varepsilon_0 \|x_k\|_X + k\|x_k\|_Z.$$

Then, $\|x_k\|_X \leq \frac{1}{\varepsilon_0}$ and $\|x_k\|_Z \leq \frac{1}{k}$ for every $k \in \mathbb{N}$. Thus, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in X and since the embedding $X \hookrightarrow Y$ is compact, there exists a subsequence $(x_k)_{k\in\Lambda\subset\mathbb{N}}$ and some $y \in Y$ such that $\|x_k - y\|_Y \to 0$ as $\Lambda \ni k \to \infty$. Since the embedding $Y \hookrightarrow Z$ is continuous, we also have $\|x_k - y\|_Z \to 0$ as $\Lambda \ni k \to \infty$. Consequently,

$$1 = \lim_{\Lambda\ni k\to\infty} \|x_k\|_Y = \|y\|_Y, \qquad \|y\|_Z = \lim_{\Lambda\ni k\to\infty} \|x_k\|_Z = 0$$

which is a contradiction.

(ii) We have the continuous embeddings

$$W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega) \hookrightarrow L^p(\Omega);$$

moreover since the domain is bounded and regular, the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ is compact as a consequence of Sobolev embedding for $p < \infty$ and of Ascoli-Arzelá for $p = \infty$. Thus for every $\varepsilon > 0$ from $(*)$ we have

$$\|u\|_{W^{2,p}(\Omega)} \leq C\, \Big(\|f\|_{L^p(\Omega)} + \varepsilon\|u\|_{W^{2,p}(\Omega)} + C_\varepsilon\|u\|_{L^p(\Omega)}\Big).$$

Choosing ε sufficiently small so that $C\varepsilon < \frac{1}{2}$, we may bring the factor $C\varepsilon\|u\|_{W^{2,p}(\Omega)}$ to the right-hand side of the inequality, and thus get

$$\frac{1}{2}\|u\|_{W^{2,p}(\Omega)} \leq C\, \Big(\|f\|_{L^p(\Omega)} + C_\varepsilon\|u\|_{L^p(\Omega)}\Big),$$

which then yields to the thesis. □

Exercise 8.3 (Sobolev inequality via Potential Theory) Let $u \in C_c^\infty(\mathbb{R}^n)$ be a fixed test function and let $\Omega \subseteq \mathbb{R}^n$ be a bounded, regular domain containing its support.

(i) Prove the following representation formula:

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \Big\langle \nabla u(y), \frac{x-y}{|x-y|^n} \Big\rangle dy = \frac{1}{\omega_{n-1}} \sum_{i=1}^n \int_{\mathbb{R}^n} \partial_i u(y) \frac{(x_i - y_i)}{|x-y|^n} dy,$$

where ω_{n-1} is the area of the unit sphere S^{n-1}. Deduce that one may estimate

$$|u(x)| \leq C \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy, \quad x \in \mathbb{R}^n.$$

(ii) Let now $1 \leq p < n$ and $p^* = \frac{np}{n-p}$ be its Sobolev conjugate. Use Young's inequality for convolution deduce from (i) that, for every $1 \leq q < p^*$, we have

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C_{\Omega,p,q} \|\nabla u\|_{L^p(\mathbb{R}^n)},$$

where $C_{\Omega,p,q}$ is a constant depending on Ω, p and q.

Remark. The Sobolev inequality deduced in (ii) is weaker than the usual one: we do not reach the sharp exponent p^* and moreover the domain Ω has to be bounded. However using some tools from functional and Fourier analysis, one may remove such limitations (see e.g. the 2^{nd} chapter of Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970).

Solution. (i) Since u is smooth, we have

$$\int_{\mathbb{R}^n} \Big\langle \nabla u(y), \frac{x-y}{|x-y|^n} \Big\rangle \, dy = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \Big\langle \nabla u(y), \frac{x-y}{|x-y|^n} \Big\rangle \, dy.$$

Integrating by parts and observing that $\operatorname{div}\left(\frac{x-y}{|x-y|^n}\right) = 0$ for $y \neq x$, we find that the right hand side equals

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(x)} \frac{u(y)}{|x-y|^{n-1}} \, d\sigma(y) = \omega_{n-1} u(x).$$

(One may argue in different ways also. For instance, one can use polar coordinates: for fixed $x \in \mathbb{R}^n$, write its polar representation $x = |x|\vartheta_x$, where $\vartheta_x \in S^n$. Then from the fundamental theorem of calculus we have

$$u(x) = \int_0^\infty \frac{\partial}{\partial r} u((|x| - r)\vartheta_x) dr = \int_0^\infty \Big\langle \nabla u((|x| - r)\vartheta_x), \vartheta_x \Big\rangle dr,$$

and now integrating with respect to ϑ_x over S^{n-1}, and then changing back variables to the Eucludean ones gives the result.

One may also start from representation formula

$$u(x) = \int_{\mathbb{R}^n} (-\Delta u)(y)\Phi(x-y)dy = \int_{\mathbb{R}^n} \Big\langle \nabla u(y), \nabla\Phi(x,y)\Big\rangle dy,$$

and integrate by parts as above shown.)

The inequality with the absolute value is then obvious.

(ii) We only need to prove the case $q > p$, the other ones are true by Hölder's inequality. As is well-known, the function

$$y \mapsto \frac{1}{|y|^{n-1}}$$

is in $L^r_{\text{loc}}(\mathbb{R}^n)$ for every $1 \leq r < \frac{n}{n-1}$. Hence, it suffices to note that one may write

$$\int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy = \int_{\Omega} \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy = |\nabla u| * \left(\frac{1}{|\cdot|^{n-1}}\chi_\Omega\right)(x),$$

and thus, from Young's inequality, we may estimate the right-hand side as

$$\begin{aligned}\left\| |\nabla u| * \left(\frac{1}{|\cdot|^{n-1}}\chi_\Omega\right)\right\|_{L^q(\Omega)} &\leq \|\nabla u\|_{L^p(\Omega)} \left\|\left(\frac{1}{|\cdot|^{n-1}}\chi_\Omega\right)\right\|_{L^r(\mathbb{R}^n)} \\ &= C_{\Omega,p,q}\|\nabla u\|_{L^p(\Omega)},\end{aligned}$$

where for any fixed $p < q < p^*$ we choose $1 \leq r < \frac{n}{n-1}$ so that

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r},$$

which is possible since

$$1 > 1 + \frac{1}{q} - \frac{1}{p} > 1 + \left(\frac{1}{p} - \frac{1}{n}\right) - \frac{1}{p} = 1 - \frac{1}{n}.$$

In particular then

$$\|u\|_{L^q(\Omega)} \leq C_{\Omega,p,q}\|\nabla u\|_{L^p(\Omega)}.$$

□

Exercise 8.4 Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $1 \leq p < \infty$. By definition the dual of $W_0^{1,p}(\Omega)$ is denoted as

$$W^{-1,p'}(\Omega) := (W_0^{1,p}(\Omega))^*,$$

where p' is the conjugate exponent of p.

(i) Prove that any $(n+1)$-tuple of functions in $L^{p'}(\Omega)$

$$f = (\psi_0, \psi_1, \ldots, \psi_n) \in L^{p'}(\Omega)^{n+1} = L^{p'}(\Omega) \times \cdots \times L^{p'}(\Omega)$$

can be identified with an element of $W^{-1,p'}(\Omega)$ via

$$(f, \varphi) = \int_\Omega \Big(\psi_0 \varphi_0 + \sum_{i=1}^n \psi_i \, \partial_i \varphi_i\Big) dx.$$

(ii) Prove the converse, namely that to every functional $f \in W^{-1,p'}(\Omega)$ it is possible to associate a $(n+1)$-tuple as in (i).

(iii) Let

$$C : L^{p'}(\Omega)^{n+1} \to W^{-1,p'}(\Omega),$$

be the correspondence found in (ii). Then C is linear and surjective but *not* injectve i.e. $\ker C \neq 0$.

Find the condition for two $(n+1)$-tuples that define the same functional in $W^{-1,p'}(\Omega)$.

Solution. (i) By Hölder's inequality we have

$$|(f, \varphi)| \leq \Big(\sum_{i=0}^n \|\psi_i\|_{L^{p'}(\Omega)}\Big) \|\varphi\|_{W^{1,p}(\Omega)} \quad \forall \, \varphi \in W^{1,p}(\Omega),$$

and hence the linear functional f so defined is bounded with

$$\|f\|_{W^{-1,p'}(\Omega)} \leq \sum_{i=0}^n \|\psi_i\|_{L^{p'}(\Omega)}.$$

(ii) Through the isometric immersion

$$\iota : W_0^{1,p}(\Omega) \to L^p(\Omega)^{n+1}, \quad \iota(\varphi) = (\varphi, \partial_1 \varphi, \ldots, \partial_n \varphi).$$

we may identify any element of $W^{-1,p'}(\Omega)$ as a continuous linear functional over $\iota(W_0^{1,p}(\Omega)) \subset L^p(\Omega)^{n+1}$.

Recall now that by Riesz repserentation theorem, the dual of $L^p(\Omega)^{n+1}$ is canonically isomorphic to $L^{p'}(\Omega)^{n+1}$ by means of integration Then, using the Hahn-Banach theorem, if $g \in (\iota(W_0^{1,p}(\Omega)))^*$, then there exists a linear and continuous extension $G \in L^{p'}(\Omega)^{n+1}$ of g, with the same norm.

This means that there exists a unique $(n+1)$-tuple of functions $\psi_i \in L^{p'}(\Omega)$ so that

$$G(u) = \int_\Omega \Big(\sum_{i=1}^n \psi_i u_i \Big) dx \quad \text{for every } (u_0, \dots u_{n+1}) \in L^p(\Omega)^{n+1}.$$

With the identification given by ι, we get the representation

$$g(\varphi) = (g, \varphi) = \int_\Omega \Big(\psi_0 \varphi_0 + \sum_{i=1}^n \psi_i \, \partial_i \varphi \Big) dx \quad \text{for } \varphi \in W_0^{1,p}(\Omega).$$

(iii) Since the map C is linear it is enough to identify its kernel. Namely, if the zero functional is represented as in (ii) we have

$$\int_\Omega \Big(\psi_0 \varphi_0 + \sum_{i=1}^n \psi_i \, \partial_i \varphi_i \Big) dx = 0 \quad \forall \, \varphi \in W_0^{1,p}(\Omega),$$

a condition that, if we set $\vec{\psi} := (\psi_1, \dots, \psi_n)$, we write concisely as follows:

$$\text{“}\psi_0 = \operatorname{div} \vec{\psi} \quad \text{in } \Omega\text{”}$$

(note that this is true in the usual sense if the functions ψ's are regular enough). With this convention, it is

$$\ker C = \Big\{ \Big(\psi_0, \vec{\psi} \Big) \in L^{p'}(\Omega)^{n+1} : \text{“}\psi_0 = \operatorname{div} \vec{\psi} \quad \text{in } \Omega\text{”} \Big\}.$$

Thus, two $(n+1)$-tuples $(\psi_0, \vec{\psi}), (\chi_0, \vec{\chi}) \in L^{p'}(\Omega)^{n+1}$ represent the same element in $W^{-1,p}(\Omega)$ via the map C if and only if

$$\text{“}\psi_0 - \chi_0 = \operatorname{div}(\vec{\psi} - \vec{\chi}) \quad \text{in } \Omega\text{”}$$

that is

$$\int_\Omega \Big(\psi_0 \varphi_0 + \sum_{i=1}^n \psi_i \, \partial_i \varphi_i \Big) dx = \int_\Omega \Big(\chi_0 \varphi_0 + \sum_{i=1}^n \chi_i \, \partial_i \varphi_i \Big) dx \quad \forall \, \varphi \in W_0^{1,p}(\Omega).$$

□

Hints to Exercises.

8.1 For (ii), recall that

- any function $u \in W_0^{1,p}(\Omega)$ can be extended *by zero* to a function $\overline{u} \in W^{1,p}(\mathbb{R}^n)$;
- the space $W^{1,p}(\mathbb{R}^n)$ is reflexive for $1 < p < \infty$;
- the embedding $W^{1,p}(B_R) \hookrightarrow L^p(B_R)$ is compact for any $R > 0$.

Distinguish the cases $p < n$ and $p = n$ and apply Sobolev's embedding.

8.2 Argue by contradiction.

8.3 For (i), you may argue similarly as in the proof of the representation formula with the fundamental solution of the Laplace equation, $u = (-\Delta u) * \Phi$ or use polar coordinates and write u as the integral of its radial derivative.

For (ii), apply carefully Young's inequality for convolution, using the fact that Ω is bounded.

8.4 For (ii), embed isometrically, $W_0^{1,p}(\Omega)$ into $L^p(\Omega)^{n+1}$, then use the well-known characterization $L^p(\Omega)* = L^{p'}(\Omega)$ and the Hahn-Banach theorem.

For (iii), compute the kernel of C.