Exercise 9.1 Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \le p < \infty$ and $\lambda > 0$. Recall that the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ is the subset of $L^p(\Omega)$ consisting of functions whose norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{\mathcal{L}^{p,\lambda}},$$

is finite, where

$$[u]_{\mathcal{L}^{p,\lambda}(\Omega)} = \sup_{\substack{x_0\in\Omega,\\ 0<r<r_0}} r^{-\frac{\lambda}{p}} \|u - u_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))},$$

with $r_0 = \min\{1, \operatorname{diam}(\Omega)\}$ and $u_{x_0,r} = \frac{1}{|\Omega\cap B_r(x_0)|}\int_{\Omega\cap B_r(x_0)} u\,dx$.

(i) Prove that $\mathcal{L}^{p,\lambda}(\Omega)$ is Banach.

(ii) Let now $r_0' > 0$ be fixed and suppose that, in the definition above, r_0 is replaced by r_0'. Prove that the corresponding Campanato norm $\|\cdot\|'_{\mathcal{L}^{p,\lambda}(\Omega)}$ is equivalent to the original one, namely that there is a constant $C > 0$ so that

$$\frac{1}{C}\|\cdot\|'_{\mathcal{L}^{p,\lambda}(\Omega)} \le \|\cdot\|_{\mathcal{L}^{p,\lambda}(\Omega)} \le C\|\cdot\|'_{\mathcal{L}^{p,\lambda}(\Omega)},$$

where C depends only on r_0', λ, p.

Solution. (i) Let $(u_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^{p,\lambda}(\Omega)$. In particular, $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ which is Banach, hence there exists $v \in L^p(\Omega)$ such that

$$\lim_{k\to\infty} \|u_k - v\|_{L^p(\Omega)} = 0.$$

It remains to prove $v \in \mathcal{L}^{p,\lambda}(\Omega)$ and $\lim_{k\to\infty}[u_k - v]_{\mathcal{L}^{p,\lambda}} = 0$. Let $x_0 \in \Omega$ and $0 < r < r_0$. Since by Hölder's inequality

$$|(u_m)_{x_0,r} - v_{x_0,r}|^p = \left|\fint_{\Omega\cap B_r(x_0)} u_m - v\,dx\right|^p \le \fint_{\Omega\cap B_r(x_0)} |u_m - v|^p\,dx \xrightarrow{m\to\infty} 0,$$

we conclude that $(u_m - (u_m)_{x_0,r})$ converges to $(v - v_{x_0,r})$ in $L^p(\Omega\cap B_r(x_0))$ as $m \to \infty$. In particular,

$$\begin{aligned} r^{-\frac{\lambda}{p}}\|v - v_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))} &= \lim_{m\to\infty} r^{-\frac{\lambda}{p}}\|u_m - (u_m)_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))} \\ &\le \limsup_{m\to\infty} [u_m]_{\mathcal{L}^{p,\lambda}}. \end{aligned} \tag{1}$$

Since $(u_m)_{m\in\mathbb{N}}$ being Cauchy in $\mathcal{L}^{p,\lambda}(\Omega)$ implies that (1) is finite, and since $x_0 \in \Omega$ and $0 < r < r_0$ are arbitrary, $[v]_{\mathcal{L}^{p,\lambda}} < \infty$ follows. Hence, $v \in \mathcal{L}^{p,\lambda}(\Omega)$.

Let $\varepsilon > 0$. By assumption, there exists $N_\varepsilon \in \mathbb{N}$ such that $[u_n - u_m]_{\mathcal{L}^{p,\lambda}} < \varepsilon$ for all $n, m \geq N_\varepsilon$ which implies that for every $x_0 \in \Omega$ and for all $0 < r < r_0$ and $n, m \geq N_\varepsilon$

$$r^{-\frac{\lambda}{p}} \|u_n - (u_n)_{x_0,r} - u_m + (u_m)_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} < \varepsilon. \tag{2}$$

As in (1), we may pass to the limit $m \to \infty$ in (2) and obtain

$$r^{-\frac{\lambda}{p}} \|u_n - (u_n)_{x_0,r} - v + v_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} < \varepsilon \tag{3}$$

for every $n \geq N_\varepsilon$. Since $x_0 \in \Omega$ and $0 < r < r_0$ are arbitrary, we conclude $[u_n - v]_{\mathcal{L}^{p,\lambda}} < \varepsilon$ for every $n \geq N_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|u_n - v\|_{\mathcal{L}^{p,\lambda}} \to 0$ as $n \to \infty$ follows.

(ii) We suppose that $r_0' < r_0$, the other case is analogous. On the one hand, the inequality

$$[u]'_{\mathcal{L}^{p,\lambda}(\Omega)} \leq [u]_{\mathcal{L}^{p,\lambda}(\Omega)}$$

is immediate, since "sup" increases as the set increases. This gives

$$\|u\|'_{\mathcal{L}^{p,\lambda}(\Omega)} \leq \|u\|_{\mathcal{L}^{p,\lambda}(\Omega)}.$$

On the other hand, observe first that, if $\omega \subset \Omega$ is any domain and $u_\omega = \frac{1}{\omega} \int_\omega d\,dx$ is the average of u over ω, we have by Hölder's inequality

$$|u_\omega| \leq \frac{1}{|\omega|} \|u\|_{L^1(\omega)} \leq \frac{1}{|\omega|^{1/p}} \|u\|_{L^p(\omega)},$$

and thus by Minkowski's inequality we can estimate

$$\|u - u_\omega\|_{L^p(\omega)} \leq \|u\|_{L^p(\omega)} + |\omega|^{1/p} |u_\omega| \leq 2\|u\|_{L^p(\omega)}. \tag{$\star$}$$

Notice now that we may write

$$[u]_{\mathcal{L}^{p,\lambda}(\Omega)} = \max\left\{ [u]'_{\mathcal{L}^{p,\lambda}(\Omega)}, \sup_{\substack{x_0 \in \Omega, \\ r_0' \leq r < r_0}} r^{-\frac{\lambda}{p}} \|u - u_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} \right\},$$

and we observe that by $(\star)$ we can estimate

$$\begin{aligned} \sup_{\substack{x_0 \in \Omega, \\ r_0' \leq r < r_0}} r^{-\frac{\lambda}{p}} \|u - u_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} &\leq \sup_{\substack{x_0 \in \Omega, \\ r_0' \leq r < r_0}} r^{-\frac{\lambda}{p}} 2\|u\|_{L^p(\Omega \cap B_r(x_0))} \\ &\leq \frac{2}{(r_0')^{\lambda/p}} \|u\|_{L^p(\Omega)}; \end{aligned}$$

hence it follows that

$$[u]_{\mathcal{L}^{p,\lambda}(\Omega)} \leq [u]'_{\mathcal{L}^{p,\lambda}(\Omega)} + C\|u\|_{\mathcal{L}^p(\Omega)},$$

where the constant C depends only on r_0', λ, p. This then yields

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} \leq C\|u\|'_{\mathcal{L}^{p,\lambda}(\Omega)},$$

and the equivalence of the norms is proved. □

Exercise 9.2 Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded of class C^1. Suppose $u \in W^{1,p}(\Omega)$ is so that

$$\mathcal{L}^n(\{x \in \Omega \mid u(x) = 0\}) \geq \alpha > 0,$$

where $\mathcal{L}^n$ denotes the Lebesgue measue on $\mathbb{R}^n$. Prove that there holds

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

for some constant $C = C(p, \alpha, n, \Omega) > 0$ independent of u.

Solution. Suppose by contradiction that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ such that, for every $k \in \mathbb{N}$,

$$\mathcal{L}^n(\{x \in \Omega \mid u_k(x) = 0\}) \geq \alpha \quad \text{and} \quad \|u_k\|_{L^p(\Omega)} > k\|\nabla u_k\|_{L^p(\Omega)}. \tag{*}$$

Up to replacing u_k with $\|u_k\|^{-1}_{L^p(\Omega)} u_k$ (an operation that preserves (*)), we may also assume $\|u_k\|_{L^p(\Omega)} = 1$. As a consequence there holds $\|u_k\|_{W^{1,p}(\Omega)} < 1 + \frac{1}{k}$ for any $k \in \mathbb{N}$ and thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$.

Since $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $1 \leq p < \infty$, possibly passing to a subsequence there exists $v \in L^p(\Omega)$ such that

$$\lim_{k\to\infty} \|u_k - v\|_{L^p(\Omega)} = 0,$$

and so, up to extracting another subsequence, also that $u_k \to v$ a.e. in Ω. Since $\|\nabla u_k\|_{L^p(\Omega)} \to 0$ as $k \to \infty$ by (*) and since the space $W^{1,p}(\Omega)$ is complete, we have $v \in W^{1,p}(\Omega)$ satisfying $\nabla v = 0$. This means that v is constant.

If we prove

$$\mathcal{L}^n(\{x \in \Omega \mid v(x) = 0\}) \geq \alpha > 0,$$

then $v \equiv 0$ would follow which would contradict $\forall k \in \mathbb{N} : \|u_k\|_{L^p(\Omega)} = 1$. Let

$$A_m = \bigcup_{k \geq m} \{x \in \Omega \mid u_k(x) = 0\},$$

$$A = \bigcap_{m=1}^{\infty} A_m.$$

Then, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and since $\mu(A_1) \leq \mu(\Omega) < \infty$ and $\mu(A_m) \geq \alpha$ we have

$$\mu(A) = \lim_{m\to\infty} \mu(A_m) \geq \alpha.$$

Since we have pointwise a.e. convergence $u_k(x) \to v(x)$ as $k \to \infty$ for almost every $x \in A$ and since by construction $u_k(x) = 0$ for infinitely many k and every $x \in A$, we conclude $v(x) = 0$ for almost every $x \in A$. Therefore,

$$\mathcal{L}^n(\{x \in \Omega \mid v(x) = 0\}) \geq \mu(A) \geq \alpha,$$

and thus the contradiction is reached. □

Exercise 9.3 Given $k \in \mathbb{N}$, let $\Omega_k = Q_+ \cup A_k \cup Q_- \subset \mathbb{R}^2$, where

$$\begin{aligned} Q_+ &= \,]1,3[\, \times \,]-1,1[, \\ A_k &= [-1,1] \times \,]-\tfrac{1}{k}, \tfrac{1}{k}[, \\ Q_- &= \,]-3,-1[\, \times \,]-1,1[. \end{aligned}$$

Denote $u_{\Omega_k} = ⨍_{\Omega_k} u \, dx$ and let $C_k = C(\Omega_k, p) \in \mathbb{R}$ be the best constant in the Poincaré inequality:

$$\int_{\Omega_k} |u - u_{\Omega_k}|^p \, dx \leq C(\Omega_k) \int_{\Omega_k} |\nabla u|^p \, dx$$

for every $u \in W^{1,p}(\Omega_k)$. Prove that $C(\Omega_k) \to \infty$ as $k \to \infty$.

Solution. For $k \in \mathbb{N}$ let $\Omega_k = Q_+ \cup A_k \cup Q_-$ and $u\colon \Omega_k \to \mathbb{R}$ be given by

$$\begin{aligned} Q_+ &= \,]1,3[\, \times \,]-1,1[, \\ A_k &= [-1,1] \times \,]-\tfrac{1}{k}, \tfrac{1}{k}[, \\ Q_- &= \,]-3,-1[\, \times \,]-1,1[, \end{aligned} \qquad u(x_1,x_2) = \begin{cases} 1, & \text{if } (x_1,x_2) \in Q_+, \\ x_1, & \text{if } (x_1,x_2) \in A_k, \\ -1, & \text{if } (x_1,x_2) \in Q_-. \end{cases}$$

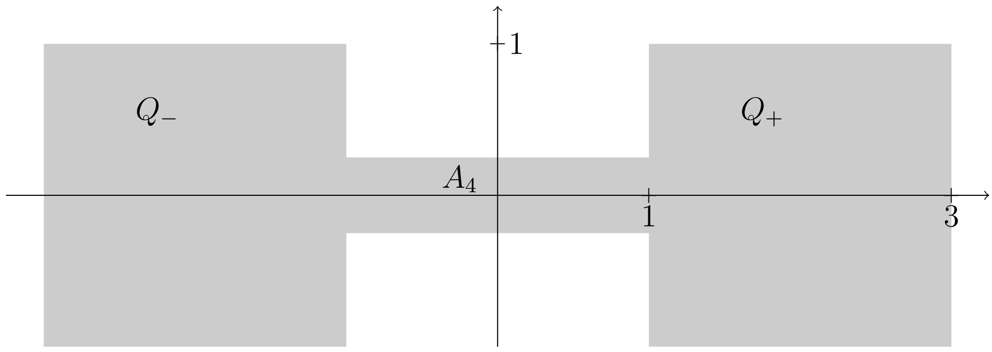
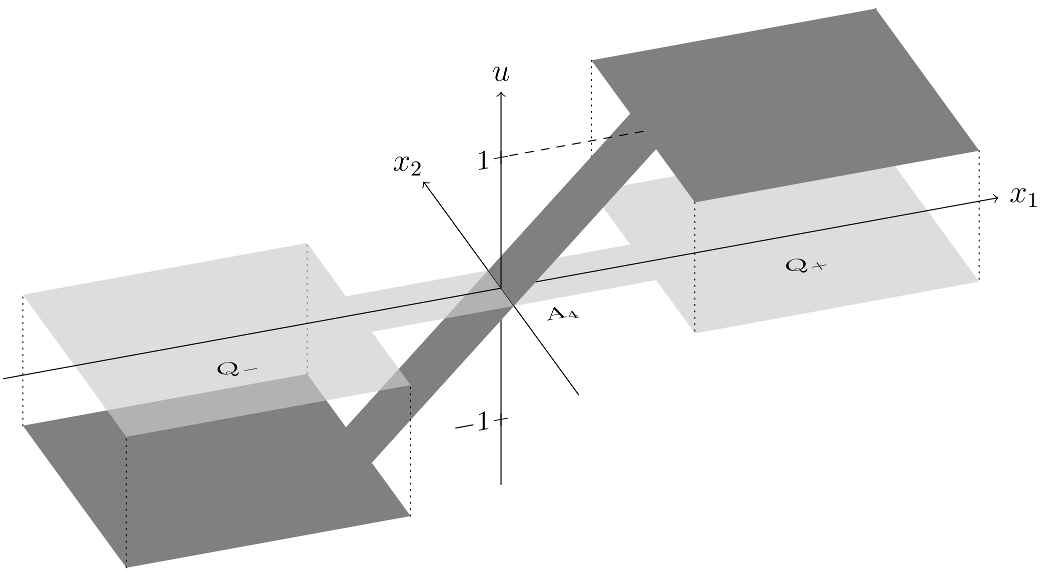


Figure 1: The domain Ω_k for $k = 4$.

Since u is Lipschitz continuous, $u \in W^{1,\infty}(\Omega)$ and because Ω is bounded $u \in W^{1,p}(\Omega)$ for any $1 \le p < \infty$. Moreover, $u_{\Omega_k} = \fint_{\Omega_k} u\,dx = 0$ and

$$\int_{\Omega_k} |u - u_{\Omega_k}|^p\,dx = \int_{\Omega_k} |u|^p\,dx \ge 8, \qquad \int_{\Omega_k} |\nabla u|^p\,dx = \int_{A_k} 1\,dx = \frac{4}{k}.$$

Combining these two facts with the assumed Poincaré inequality, we have $8 \le C_k \frac{4}{k}$. Therefore, $C_k \ge 2k \to \infty$ as $k \to \infty$. □



Exercise 9.4 Let Ω be $\mathbb{R}^n$ or a bounded domain with regular boundary. Suppose you know the validity of the inclusions:

$$W^{1,p}(\Omega) \subset L^{p*}(\Omega) \quad (1 \le p < n)$$
$$W^{1,p}(\Omega) \subset C^{0,\alpha}(\overline{\Omega}) \quad (n < p < \infty),$$

where $p^* = \frac{np}{n-p}$, $\alpha = 1 - \frac{n}{p}$, but *only* in a set-theoretic sense, without information on the topologies (for the second case, we adopt the usual convention i.e. we suppose a unique continuous representative has been identified).

Prove that this is enough to conclude that such inclusions are in fact continuous embeddings.

Solution. We treat the first case, the second is analogous. From the assumed inclusion $W^{1,p}(\Omega) \subset L^{p*}(\Omega)$ we can consider the identity map

$$\iota : W^{1,p}(\Omega) \subset L^{p*}(\Omega), \quad \iota(x) = x.$$

which is clearly linear. Since both the spaces in question are Banach, by the Closed Graph Theorem it is enough to prove that ι has closed graph.

Let $(u_k, u_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,p}(\Omega) \times L^{p*}(\Omega)$ converging in the product topology to some (u, g). If $\omega \subset\subset \Omega$ is any bounded domain with compact closure in Ω, $W^{1,p}$ and L^{p*} convergence imply in particular convergence in $L^1(\omega)$, and so by the uniqueness of the limit, we have

$$u = g \quad \text{in } L^1(\omega) \text{ and a.e. in } \omega,$$

but since ω is arbitrary, we deduce that $f = g$. Thus the graph of ι is closed and hence the inclusion is actually an immersion. □

Exercise 9.5 Let $n < p < \infty$.

(i) Prove that for any $u \in W^{1,p}(\mathbb{R}^n)$, there holds (for its continuous representative)

$$\lim_{x\to\infty} u(x) = 0.$$

(ii) It is possible to quantify the decay of u at infinity, namely, is it possible find some $\beta = \beta(n, p) > 0$ so that

$$\text{“}u(x) = O\left(\frac{1}{|x|^\beta}\right) \quad \text{as } x \to \infty\text{”}?$$

(iii) How does the answer to (ii) change if we additionally suppose $u \in W^{k,p}(\mathbb{R}^n)$ for $k = 2, 3, \ldots$?

Solution. (i) Let $(u_j)_{j\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n)$ be an approximating sequence for u in $W^{1,p}(\mathbb{R}^n)$:

$$\lim_{j\to\infty} \|u_j - u\|_{W^{1,p}(\mathbb{R}^n)} = 0;$$

by Sobolev embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$ it follows

$$\lim_{j\to\infty} \|u_j - u\|_{C^{0,\alpha}(\mathbb{R}^n)} = 0.$$

Fix now $\varepsilon > 0$ and let $j \in \mathbb{N}$ be so that $\|u - u_j\|_{L^\infty(\mathbb{R}^n)} < \varepsilon$ and let $R_j > 0$ be sufficiently big so that $\operatorname{supp}(u_j) \subset B_{R_j}(0)$. For any $x \in \mathbb{R}^n \setminus \overline{B_{R_j}(0)}$ we then see that:

$$|u(x)| = |u(x) - u_j(x)| \le \varepsilon,$$

and, since ε is arbitrary, this gives $\lim_{x\to\infty} u(x) = 0$.

(ii)–(iii) No general quantitative statement as above requested is possible, as the following construction demonstrates. Let $\phi \in C_c^\infty(B_1(0))$ be any smooth function with $\phi(0) = 1$ and consider the function:

$$u(x) = \sum_{j=1}^{\infty} \frac{1}{2^j}\phi\Big(x - 2^{j^2} e_1\Big), \quad x \in \mathbb{R}^n,$$

where $e_1 = (1, 0, \ldots, 0)$ is the first vector of the canonical basis of $\mathbb{R}^n$. This function is smooth, vanishes at infinity and belongs to any Sobolev space $W^{k,p}(\mathbb{R}^n)$, for any $k \in \mathbb{N}$ and any $p \in [1, \infty]$. However there is no $\beta > 0$ for which $\lim_{x\to\infty} |x|^\beta u(x)$ is finite. Indeed, since

$$|x|^\beta \phi\Big(x - 2^{j^2}\Big) \neq 0 \quad \text{if and only if} \quad x \in B_1\left(2^{j^2} e_1\right),$$

then, for any $\beta > 0$ choosing $x = 2^{j^2}$ we see that for any $j \in \mathbb{N}$.

$$|x|^\beta u(x) = \frac{2^{\beta j^2}}{2^j}\phi(0) = 2^{\beta j^2 - j},$$

so in particular for no $\beta > 0$ it can be $u(x) = O(1/|x|^\beta)$ as $x \to \infty$. □

Hints to Exercises.

9.2 Argue by contradiction similarly as in the proof of the Poincaré inequality for $W_0^{1,p}$.

9.4 Use a suitable theorem of Functional Analysis I.

9.5 For (i), argue by approximation and use Sobolev embedding.

For (ii), construct suitably a function consisting of infinitely many smooth "bumps" that get smaller and smaller as you approach infinity...