Exercise 10.1

(i) Prove that for every $\alpha \in (0,1)$ we have the embedding $W^{2,n}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$.

(ii) Prove that, in general, a function in $W^{2,n}(\mathbb{R}^n)$ needs *not* to be Lipschitz continuous.

Solution. (i) Both u and ∇u are in $W^{1,n}(\mathbb{R}^n)$ which embeds in $L^p(\mathbb{R}^n)$ for every $p \in [n, \infty)$. For $0 < \alpha < 1$ we may then choose $p = \frac{n}{1-\alpha}$ and conclude via the embedding $W^{1,\frac{n}{1-\alpha}}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$.

(ii) Consider the function

$$u(x) := \begin{cases} x_1 \log|\log|x||, & \text{if } x \in B_{\frac{1}{2}}(0) \setminus \{0\}, \\ 0, & \text{if } x = 0. \end{cases}$$

Since

$$\frac{\partial}{\partial x_1} u(x) = \log|\log|x|| + O\left(\frac{1}{\log|x|}\right) \quad \text{as } x \to 0,$$

u is not Lipschitz.

Let us show that $u \in W^{2,n}(B_{1/2}(0))$. Set for brevity

$$f(x) = \log|\log|x||.$$

First derivatives (away from 0) are given by

$$\frac{\partial u}{\partial x_i}(x) = \delta_{1,i} f(x) + x_1 \frac{\partial f}{\partial x_i}(x),$$

with

$$\frac{\partial f}{\partial x_i}(x) = \frac{x_i}{(\log|x|)|x|^2}.$$

Similarly second derivatives are given by

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \delta_{1,i}\left(\frac{x_j}{(\log|x|)|x|^2}\right) + \delta_{1,j} \frac{\partial f}{\partial x_i}(x) + x_1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

with

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \delta_{i,j} \frac{1}{(\log|x|)|x|^2} - \frac{x_i}{(\log|x|)^2|x|^4}(x_j + 2(\log|x|)x_j).$$

For $C > 0$ large enough, we can estimate

$$|\nabla^2 u(x)| \leq \frac{C}{|x||\log|x||}$$

and we can compute that

$$\int_{B_{1/2}(0)} \left(\frac{1}{|x||\log|x||}\right)^n dx = C\int_0^{\frac{1}{2}} \frac{dr}{r(\log r)^n} < \infty.$$

Therefore, for $n \geq 2$ we have that the classical derivatives satisfy

$$|\nabla u|, |\nabla^2 u| \in L^n(B_{1/2}(0)).$$

Since $\operatorname{cap}_{W^{1,p}}(\{0\}) = 0$ for $p = \frac{n}{n-1} \leq n$ (Beispiel 8.1.1) the expressions above coincide with the the weak first and second derivatives of u in $B_{1/2}(0)$ (by Satz 8.1.1). Thus $W^{2,n}(B_{1/2}(0))$.

To obtain an example in $W^{2,n}(\mathbb{R}^n)$, pick an extension of u in this space. $\square$

Exercise 10.2

(i) Prove that there is a continuous embedding

$$W^{n,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n).$$

(ii) Are functions in $W^{n,1}(\mathbb{R}^n)$ also cointinuous (i.e. have a continuous representative)?

Solution. (i) Let $u \in C_c^\infty(\mathbb{R}^n)$ and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be arbitrary. Then,

$$\begin{aligned} u(x_1, \ldots, x_n) &= \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(s_1, x_2, \ldots, x_n)\, ds_1 \\ &= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{\partial^2 u}{\partial x_2 \partial x_1}(s_1, s_2, x_2, \ldots, x_n)\, ds_2\, ds_1 \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{\partial^n u}{\partial x_n \ldots \partial x_1}(s_1, \ldots, s_n)\, ds_n \ldots\, ds_1, \end{aligned}$$

hence

$$|u(x)| \leq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left|\frac{\partial^n u}{\partial x_n \ldots \partial x_1}(s_1, \ldots, s_n)\right| ds_n \ldots\, ds_1 \leq \|u\|_{W^{n,1}(\mathbb{R}^n)}.$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have deduced

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{W^{n,1}(\mathbb{R}^n)}. \tag{*}$$

The inequality (*) remains true for arbitrary $u \in W^{n,1}(\mathbb{R}^n)$ by density of $C_c^\infty(\mathbb{R}^n)$ in $W^{n,1}(\mathbb{R}^n)$. Indeed, given $u \in W^{n,1}(\mathbb{R}^n)$, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^n)$ such that $\|u_k - u\|_{W^{n,1}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Since inequality (*) implies $\|u_k - u_m\|_{L^\infty(\mathbb{R}^n)} \leq \|u_k - u_m\|_{W^{n,1}(\mathbb{R}^n)}$ the sequence $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $L^\infty(\mathbb{R}^n)$ and hence convergent to some v in $L^\infty(\mathbb{R}^n)$. Up to a subsequence, we also have that $u_k(x) \to v(x)$ converges for a.e. $x \in \mathbb{R}^n$ But since $\|u_k - u\|_{L^n(\mathbb{R}^n)} \to 0$ also implies pointwise convergence almost everywhere on a subsequence, $v = u$ almost everywhere follows by uniqueness of limits. Passing to the limit $k \to \infty$ in $\|u_k\|_{L^\infty(\mathbb{R}^n)} \leq \|u_k\|_{W^{n,1}(\mathbb{R}^n)}$ proves the claim.

(ii) Yes, since if u is a function in $W^{n,1}(\mathbb{R}^n)$, and $(u_k)_k \subseteq C_c^\infty(\mathbb{R}^n)$ is a sequence approximating it, then by (i) the limit is also uniform. Hence u is also continuous (i.e. has a continuous representative). □

Exercise 10.3 Recall that an *algebra* is a vector space V endowed with a bilinear operation "$\cdot \times \cdot$" $: S \times S \to S$.

Let $\Omega \subseteq \mathbb{R}^n$ be a regular domain.

(i) Prove that whenever $p > n$, $W^{1,p}(\Omega)$ is an algebra with respect to the usual multiplication between functions:

$$W^{1,p}(\Omega) \times W^{1,p}(\Omega) \to W^{1,p}(\Omega), \quad f \times g = fg,$$

and moreover that such multiplication is continuous.

(ii) Prove that the statement is false when $p = n$. What is the biggest subspace $X \subset W^{1,n}(\mathbb{R}^n)$ that comes to your mind, where the multiplication becomes an algebra? Try also to find a norm for this space where the multiplication is continuous.

Solution. (i) By the Sobolev-Morrey embedding we have $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ continuously. Consequently for $f, g \in W^{1,p}(\Omega)$ with the product rule for weak derivaties we can estimate

$$\begin{aligned}
&\|fg\|_{W^{1,p}} \\
&\leq C\Big(\|f\, g\|_{L^p} + \|\nabla f\, g\|_{L^p} + \|f\, \nabla g\|_{L^p}\Big) \\
&\leq C\Big(\|f\|_{L^\infty}\|g\|_{L^p} + \|\nabla f\|_{L^p}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\nabla g\|_{L^p}\Big) \\
&\leq C\|f\|_{W^{1,p}}\|g\|_{W^{1,p}},
\end{aligned}$$

which also proves continuity.

(ii) The statement is false, since the product of a function in $L^n(\Omega)$ and one in $W^{1,n}(\Omega)$ is not in $L^n(\Omega)$, we cannot estimate as above $\|f\nabla g\|_{L^n(\Omega)}$ and $\|\nabla f\, g\|_{L^n(\Omega)}$.

However $W^{1,n}(\Omega) \hookrightarrow \bigcap_{1\le p<\infty} L^p(\Omega)$, so it misses $L^\infty(\Omega)$ by very little. And in fact if we consider $X = W^{1,n}(\Omega) \cap L^\infty(\Omega)$ with norm $\|\varphi\|_X = \|\varphi\|_{W^{1,n}} + \|\varphi\|_{L^\infty}$ then this subspace is an algebra, since

$$
\begin{aligned}
&\|fg\|_X \\
&\le C\Big(\|f\, g\|_{L^n} + \|\nabla f\, g\|_{L^n} + \|f\, \nabla g\|_{L^n} + \|fg\|_{L^\infty}\Big) \\
&\le C\Big(\|f\|_{L^\infty}\|g\|_{L^n} + \|\nabla f\|_{L^n}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\nabla g\|_{L^n} + \|f\|_{L^\infty}\|g\|_{L^\infty}\Big) \\
&\le C\|f\|_X\|g\|_X.
\end{aligned}
$$

□

Exercise 10.4 (Caccioppoli's Inequality) Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $u \in W^{1,2}(\Omega)$ be a weak solution to Laplace's equation,

$$-\Delta u = 0 \quad \text{in } \Omega,$$

namely

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle\, dx = 0 \quad \forall\, \varphi \in W_0^{1,2}(\Omega). \tag{$\triangle$}$$

(i) Let $x_0 \in \Omega$ and let $R > 0$ so that $B_R(x_0) \subset \Omega$. Chose suitably a test function φ in ($\triangle$) to prove that, for every $\rho \in (0, R)$ and every $\lambda \in \mathbb{R}$ the following inequality holds:

$$\int_{B_\rho(x_0)} |\nabla u|^2\, dx \le \frac{C}{(R-\rho)^2}\int_{B_R(x_0)} |u-\lambda|^2\, dx,$$

where $C > 0$ is a universal constant.

(ii) Find the minimum of the function

$$f(\lambda) = \int_{B_R(x_0)} |u-\lambda|^2\, dx, \quad \lambda \in \mathbb{R}.$$

Solution. (i) We choose as a test function

$$\varphi = \eta^2(u-\lambda),$$

where $\eta \in C_c^\infty(B_R(x_0))$ is so that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_r(x_0)$ and $|\nabla \eta| \le \frac{C}{R-\rho}$. We get

$$\int_{B_R(x_0)} |\nabla u|^2 \eta^2 dx + \int_{B_R(x_0)} \langle \nabla u, 2(u-\lambda)\eta(\nabla \eta)\rangle dx = 0.$$

Now, on the one hand, we may estimate with Hölder's inequality

$$\begin{aligned}
&\left| \int_{B_R(x_0)} \langle \nabla u, 2(u-\lambda)\eta(\nabla \eta)\rangle dx \right| \\
&\le \frac{C}{R-\rho} \int_{B_R(x_0)} |\nabla u||u-\lambda|\eta \, dx \\
&\le \frac{C}{R-\rho} \Big(\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, dx \Big)^{1/2} \Big(\int_{B_R(x_0)} |u-\lambda|^2 \, dx \Big)^{1/2},
\end{aligned}$$

consequently

$$\int_{B_R(x_0)} |\nabla u|^2 \eta^2 dx \le \frac{C}{R-\rho} \Big(\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, dx \Big)^{1/2} \Big(\int_{B_R(x_0)} |u-\lambda|^2 \, dx \Big)^{1/2},$$

and thus dividing by $\left(\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, dx\right)^{1/2}$ (or using Young's inequality) yields

$$\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, dx \le \frac{C}{(R-\rho)^2} \int_{B_R(x_0)} |u-\lambda|^2 \, dx;$$

finally using

$$\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, dx \ge \int_{B_\rho(x_0)} |\nabla u|^2 \, dx$$

yields the estimate.

(ii) Differentiating f with respect to λ gives

$$f'(\lambda) = -2 \int_{B_R(x_0)} (u-\lambda) dx,$$

and thus, the only critical point of f is the average of u:

$$\bar{\lambda} = u_{x_0,r} = ⨍_{B_R(x_0)} u \, dx.$$

Since f is non negative and $f''(\lambda) = 2|B_R(x_0)| > 0$, $\overline{\lambda}$ is its absolute minimum.

□

Exercise 10.5 Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(i) Prove that

$$[u, v] = \int_\Omega \Delta u \Delta v \, dx$$

defines a scalar product on $H^2_0(\Omega)$ equivalent to the standard one $(\cdot, \cdot)_{H^2(\Omega)}$.

(ii) Prove that for every $f \in L^2(\Omega)$ there is a unique $u \in H^2_0(\Omega)$ satisfying

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \quad \text{for every } v \in H^2_0(\Omega)$$

(iii) Let $f \in L^2(\Omega)$. Find the boundary value problem for which u found in (ii) is a weak solution and prove that it is the unique weak solution of class H^2 for such problem.

Solution. (i) Symmetry and bilinearity are immediate.

Note now that for every $u \in C^\infty_c(\Omega)$ we have

$$\int_\Omega |\Delta u|^2 \, dx = \int_\Omega |\nabla^2 u|^2 \, dx. \tag{1}$$

With Poincaré's inequality we get that

$$\int_\Omega |u|^2 \mathrm{d}x \leq C \int_\Omega |\nabla u|^2 \mathrm{d}x \leq C \int_\Omega |\nabla^2 u|^2 \mathrm{d}x. \tag{2}$$

By definition of $H^2_0(\Omega)$ we can approximate every $u \in H^1_0(\Omega)$ by functions in $C^\infty_c(\Omega)$. Therefore, (1) and (2) are also true for functions $u \in H^2_0(\Omega)$. This gives

$$\|u\|^2_{H^2(\Omega)} \leq C[u, u].$$

On the other hand we have that

$$[u, u] = \int_\Omega |\nabla^2 u|^2 \mathrm{d}x \leq \|u\|^2_{H^2_0(\Omega)}.$$

So $[\cdot, \cdot]$ is positive definite and it is equivalent to the standard scalar product on $H^2_0(\Omega)$ (and in particular its topology is Hilbert).

(ii) Let $f \in L^2(\Omega)$. The function

$$\beta : H^2_0(\Omega) \to \mathbb{R} \quad \beta(v) = \int_\Omega f v \, dx$$

is an element of $(H_0^2(\Omega))^*$ and by (i), $[\cdot,\cdot]$ is equivalent to the standard scalar product. So by Riesz' Represention Theorem there exists a unique $u \in H_0^2(\Omega)$ with

$$\int_\Omega fv\,dx = [u,v],$$

for every $v \in H_0^2(\Omega)$.

(iii) We claim that the $u \in H_0^2(\Omega)$ we found in (ii) is the unique weak solution of the following boundary value problem:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta^2 u = \Delta(\Delta u)$ and $\partial_\nu u = \langle \nabla u, \nu\rangle$ is the (outward) normal derivative across $\partial\Omega$. Indeed, for every $\varphi \in C_c^\infty(\Omega)$ we have that

$$\int_\Omega f\varphi\,dx = [u,\varphi] = \int_\Omega \Delta u \Delta\varphi\,dx = \int_\Omega u\Delta^2\varphi\,dx,$$

so u is a weak solution of $\Delta^2 u = f$ in Ω.

Note next that since $u \in H_0^2(\Omega)$ both the traces $u|_{\partial\Omega}$ and $\nabla u|_{\partial\Omega}$ exist and vanish by construction. Since however trace operator $(\cdot)|_{\partial\Omega}$ and tangential derivative ∂_τ along $\partial\Omega$ commute, from the decomposition

$$\nabla u|_{\partial\Omega} = \partial_\tau u + \partial_\nu u,$$

we obtain that, in our case, $\nabla u|_{\partial_\Omega}$ vanishes if and only if $\partial_\nu u$ does.

As for uniqueness, if u_1, u_2 are two weak solutions of the above problem, then

$$[u_1, v] = [u_2, v] = \beta(v) \quad \forall\, v \in H_0^2(\Omega),$$

hence

$$[u_1 - u_2, v] = 0 \quad \forall\, v \in H_0^2(\Omega)$$

setting $v = u_1 - u_2$ and using the fact that $[\cdot,\cdot]$ is positive definite yields $u_1 - u_2 = 0$. $\square$

Hints to Exercises.

10.1 For (ii) look for a counterexample involving the logarithm, similarly as those presented in the lectures dealing with capacity (§8.1)

10.2 Modify the proof of Satz 8.5.1.

10.4 For (i), look for φ equal to u times a suitable cutoff function for $B_\rho(x_0)$. Then work your way with Hölder, Young and the properties of the cutoff you have chosen.

For (ii), look at the derivative of f.