Exercise 11.1 What follows is often useful in regularity theory.

Let $p \in [1, \infty)$.

(i) Let $\Omega \subset \mathbb{R}^n$ be a domain with finite and nonzero Lebesgue measure. Prove that there exists a constant $C = C(p, \Omega) > 0$ depending only on p so that for every $u \in L^p(\Omega)$ there holds

$$\int_\Omega |u - u_\Omega|^p \, dx \leq C \inf_{\lambda \in \mathbb{R}} \int_\Omega |u - \lambda|^p \, dx,$$

where $u_\Omega = \fint_\Omega u \, dx$ is the average of u over Ω.

(ii) Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $\lambda > 0$ and $u \in L^p_{\text{loc}}(\Omega)$. an let Ω', Ω'' be bounded domains with Ω' of Type A and

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega.$$

Fix and $R > 0$ so that $B_R(x_0) \subset \Omega''$ for every $x_0 \in \Omega'$. Suppose you know that

$$\{u\}^p_{\mathcal{L}^{p,\lambda}(\Omega')} := \sup_{\substack{x_0 \in \Omega' \\ \rho \in (0, r_0)}} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)} |u(x) - u_{B_\rho(x_0)}|^p \, dx < \infty.$$

Use (i) to prove that then u belongs to $\mathcal{L}^{p,\lambda}(\Omega')$ (the usual Campanato space) and the estimate

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega')} \leq C\Big(\|u\|_{L^p(\Omega')} + \{u\}_{\mathcal{L}^{p,\lambda}(\Omega')}\Big)$$

holds for a constant $C = C(p, \lambda, \Omega', \Omega'') > 0$ that does not depend on u.

Solution. (i) One may notice that for $p = 2$ by taking derivatives the inequality is an equality with $C = 1$. In general, using the inequalities

$$\begin{aligned}
|a + b|^p &\leq C_p(|a|^p + |b|^p), \\
|v_\Omega|^p &\leq \frac{1}{|\Omega|}\|v\|^p_{L^p(\Omega)},
\end{aligned}$$

we may estimate for every $\lambda \in \mathbb{R}$:

$$\begin{aligned}
\int_\Omega |u - u_\Omega|^p \, dx &= \int_\Omega |(u - \lambda) - (u - \lambda)_\Omega|^p \, dx \\
&\leq C_p \int_\Omega \Big(|(u - \lambda)|^p + |(u - \lambda)_\Omega|^p\Big) \, dx \\
&\leq C_p \int_\Omega |(u - \lambda)|^p \, dx,
\end{aligned}$$

thus yielding the thesis by taking the infimum in λ.

(ii) Thanks to (i), for every $x_0 \in \Omega'$ and $\rho \in (0, R)$ we may estimate

$$\begin{aligned}\int_{\Omega'(x_0,\rho)} |u - u_{\Omega'(x_0,\rho)}|^p \, dx &\leq C_p \int_{\Omega'(x_0,\rho)} |u - u_{B_\rho(x_0)}|^p \, dx \\ &\leq C_p \int_{B_\rho(x_0)} |u - u_{B_\rho(x_0)}|^p \, dx,\end{aligned}$$

whence

$$\sup_{\substack{x_0 \in \Omega' \\ \rho \in (0,R)}} \frac{1}{\rho^\lambda} \int_{\Omega'(x_0,\rho)} |u - u_{\Omega'(x_0,\rho)}|^p \, dx \leq C_p \{u\}_{\mathcal{L}^{p,\lambda}(\Omega')}.$$

Thanks to Exercise 9.1 we may then conclude that

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega')} \leq C\Big(\|u\|_{L^p(\Omega')} + \{u\}_{\mathcal{L}^{p,\lambda}(\Omega')} \Big),$$

for a constant $C = C(p, \lambda, R) = C(p, \lambda, \Omega', \Omega'') > 0$. □

Exercise 11.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded, regular domain and let $\Delta^2 \varphi = \Delta(\Delta\varphi)$ be the Bilaplacian. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and

$$\Xi := \{u \in H^4(\Omega) \cap H^1_0(\Omega) \mid \Delta u \in H^1_0(\Omega)\}.$$

(i) Prove that the Bilaplacian

$$\Delta^2 \colon \Xi \to L^2(\Omega), \quad u \mapsto \Delta(\Delta u),$$

is bijective from Ξ onto $L^2(\Omega)$.

(ii) Given $f \in L^2(\Omega)$, let $u \in \Xi$ satisfy $\Delta^2 u = f$. Prove that for every $\varphi \in \Xi$ there holds

$$\int_\Omega u \Delta^2 \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{†}$$

(iii) Assume that $u, f \in L^2(\Omega)$ satisfy (**??**). Prove that $u \in \Xi$.

Solution. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and

$$\Xi := \{u \in H^4(\Omega) \cap H^1_0(\Omega) \mid \Delta u \in H^1_0(\Omega)\}.$$

(i) Since the bilaplacian $\Delta^2 \colon \Xi \to L^2(\Omega)$ is linear, it suffices to prove $\ker(\Delta^2) = \{0\}$ to conclude injectivity. Let $u \in \Xi$ with $\Delta^2 u = 0$. By definition of Ξ, we have

$$v := \Delta u \in H^2(\Omega) \cap H^1_0(\Omega).$$

Moreover, $\Delta v = 0$ combined with the elliptic regularity estimate (Satz 9.1.2) implies $v = 0$. Repeating the same argument for $\Delta u = 0$ yields $u = 0$ and proves $\ker(\Delta^2) = 0$.

To prove surjectivity, let $f \in L^2(\Omega)$ be given arbitrarily. Let $v \in H^1_0(\Omega)$ be the weak solution to $\Delta v = f$ in Ω. By elliptic regularity, $v \in H^2(\Omega)$. Let $u \in H^1_0(\Omega)$ be the weak solution to $\Delta u = v$. Then, by elliptic regularity, $u \in H^4(\Omega)$. Consequently, $u \in \Xi$. Since $\Delta^2 u = f$ by construction, surjectivity of $\Delta^2 \colon \Xi \to L^2(\Omega)$ follows.

(ii) Let $\varphi \in \Xi$ be arbitrary. Then, $\nabla\Delta\varphi \in L^2(\Omega)$. Since $u \in H^1_0(\Omega)$, the trace theroem (Satz 8.4.3) implies that $u|_{\partial\Omega} \in L^2(\partial\Omega)$ is well-defined and vanishes according to Korollar 8.4.3. Analogously, since $\Delta\varphi \in H^1_0(\Omega)$ by assumption, $(\Delta v)|_{\partial\Omega} = 0$. Hence, we may integrate by parts twice with vanishing boundary terms to obtain

$$\int_\Omega u\Delta^2\varphi\,dx = -\int_\Omega \nabla u \cdot \nabla\Delta\varphi\,dx = \int_\Omega \Delta u \Delta\varphi\,dx. \tag{$*$}$$

Since the right hand side of $(*)$ is symmetric in u and φ we may switch the roles of $u, \varphi \in \Xi$ to also obtain

$$\int_\Omega \varphi\Delta^2 u\,dx = \int_\Omega \Delta u\Delta\varphi\,dx = \int_\Omega u\Delta^2\varphi\,dx.$$

Since $\varphi \in \Xi$ is arbitrary, the claim follows by substituting $\Delta^2 u = f$.

(iii) According to part (i), there exists $v \in \Xi$ such that $\Delta^2 v = f$. Moreover, by (ii) for every $\varphi \in \Xi$ there holds

$$\int_\Omega v\Delta^2\varphi\,dx = \int_\Omega f\varphi\,dx.$$

Therefore, using again bijectivity of $\Delta^2 \colon \Xi \to L^2(\Omega)$ as shown in (i), we have

$$\int_\Omega (u - v)\Delta^2\varphi\,dx = 0 \quad \forall\, \varphi \in \Xi$$

if and only if

$$\int_\Omega (u - v)\psi\,dx \quad \forall\, \psi \in L^2(\Omega)$$

Hence $u - v = 0$ in $L^2(\Omega)$. Therefore, $u = v \in \Xi$ as claimed. $\square$

Exercise 11.3 We revisit Exercise 10.5 with the notions ellpitic regularity theory we have acquired.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(i) Prove that

$$\langle u, v\rangle := \int_\Omega \Delta u \Delta v \, dx$$

defines a scalar product on $H^2(\Omega) \cap H^1_0(\Omega)$ which is equivalent to the standard scalar product $(\cdot,\cdot)_{H^2(\Omega)}$.

(ii) Show that $\big(H^2(\Omega) \cap H^1_0(\Omega), \langle\cdot,\cdot\rangle\big)$ is a Hilbert space.

(iii) Prove that given $f \in L^2(\Omega)$ there is a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying

$$\forall v \in H^2(\Omega) \cap H^1_0(\Omega): \quad \int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx.$$

Show that in fact $u \in \Xi := \{u \in H^4(\Omega) \cap H^1_0(\Omega) \mid \Delta u \in H^1_0(\Omega)\}$ and $\Delta^2 u = f$.

Solution. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(i) The map $\langle\cdot,\cdot\rangle$ is symmetric and bilinear by definition. Moreover, by the elliptic regularity estimate (Satz 9.1.2), there exists a constant $C < \infty$ such that for every $u \in H^2(\Omega) \cap H^1_0(\Omega)$

$$\langle u, u\rangle \leq (u,u)_{H^2(\Omega)} = \|u\|^2_{H^2(\Omega)} \leq C\|\Delta u\|^2_{L^2(\Omega)} = C\langle u, u\rangle.$$

In particular, $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$; hence $\langle\cdot,\cdot\rangle$ defines a scalar product and $\langle\cdot,\cdot\rangle$ is equivalent to $(\cdot,\cdot)_{H^2(\Omega)}$.

(ii) Since Ω is bounded, convergence in $H^2(\Omega)$ implies convergence in $H^1(\Omega)$. Since $H^1_0(\Omega)$ is closed in $H^1(\Omega)$, we obtain that $H^2(\Omega) \cap H^1_0(\Omega)$ is closed in $H^2(\Omega)$. Hence, $(H^2(\Omega) \cap H^1_0(\Omega), \langle\cdot,\cdot\rangle)$ is a Hilbert space.

(iii) Then the map $H^2(\Omega) \cap H^1_0(\Omega) \to \mathbb{R}$ given by $v \mapsto \int_\Omega f v \, dx$ is a continuous linear functional. By part (ii) we may apply the Riesz representation theorem to conclude that there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying

$$\forall v \in H^2(\Omega) \cap H^1_0(\Omega): \quad \langle u, v\rangle = \int_\Omega f v \, dx.$$

In particular, for any $v \in \Xi := \{u \in H^4(\Omega) \cap H^1_0(\Omega) \mid \Delta u \in H^1_0(\Omega)\}$,

$$\int_\Omega u \Delta^2 v \, dx = \int_\Omega \Delta u \Delta v = \int f v \, dx.$$

Hence, $u \in \Xi$ according to Exercise 11.2 and

$$\int_\Omega (\Delta^2 u) v \, dx = \int_\Omega u \Delta^2 v \, dx = \int f v \, dx$$

for any $v \in C_c^\infty(\Omega)$ which implies $\Delta^2 u = f$. □

Exercise 11.4 Let $\Omega \subseteq \mathbb{R}^n$ be a regular domain. Consider a function $u \in H_0^1(\Omega)$ so that its weak Laplacian Δu is in $L^2(\Omega)$, namely, there exists $f \in L^2(\Omega)$ so that for every $\varphi \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega u \Delta \varphi \, dx = \int_\Omega f \varphi \, dx,$$

for which we then set $f = \Delta u$. By definition, there exist a sequence $u \in C_c^\infty(\Omega)$ so that

$$\lim_{k\to\infty} u_k = u \quad \text{in } H^1(\Omega)$$

We ask whether the sequence can be chosen so that *additionally* it satisfies

$$\lim_{k\to\infty} \Delta u_k = \Delta u \quad \text{in } L^2(\Omega).$$

(i) Prove that the answer is positive when $\Omega = \mathbb{R}^n$.

(ii) Prove that the answer is, in general, negative for $u \in H_0^1(\Omega)$ when Ω is bounded.

(iii) Can you characterize the subset of functions $u \in H_0^1(\Omega)$ for which such approximating sequence exists?

Solution. (i) We construct an approximating sequence with the required properties explicitly, by first truncating and then mollifying u on progressively larger balls. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a smooth cut-off function so that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_1(0)$ and supported in $B_2(0)$ and let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier. Denoting, for $x \in \mathbb{R}^n$,

$$\eta_k(x) = \eta\Big(\frac{x}{k}\Big),$$
$$\phi_k(x) = k\,\phi(k\,x),$$

we let

$$u_k(x) = (u\eta_k) * \phi_k\,(x).$$

Then, every u_k is smooth compactly supported; moreover

$$\|u_k - u\|_{L^2(\mathbb{R}^n)} \leq \|u * \phi_k - u\|_{L^2(\mathbb{R}^n)} + \|(u - u\eta_k) * \phi_k\|_{L^2(\mathbb{R}^n)},$$

and as $k \to \infty$ both terms on the right-hand side vanish: the first one by standard properties of the convolution, the second since

$$\|(u - u\eta_k) * \phi_k\|_{L^2(\mathbb{R}^n)} \leq C\|(u - u\eta_k)\|_{L^2(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.$$

As for first derivatives, using the fact that weak derivatives and convolution commute and the product rule for weak derivatives we have

$$\partial_i u_k = (\partial_i u\, \eta_k + u\, \partial_i \eta_k) * \phi_k.$$

As before, we have

$$\partial_i u\, \eta_k * \phi_k \xrightarrow{k\to\infty} \partial_i u \quad \text{in } L^2(\mathbb{R}^n),$$

and by the properties of η_k, we see that

$$\|u\, \partial_i \eta_k * \phi_k\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{k}\|u\|_{L^2(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.$$

A similar computation holds for the Laplacian, being

$$\Delta u_k = (\Delta u\, \eta_k + 2\langle \nabla u, \nabla \eta_k\rangle + u\Delta\eta_k) * \phi_k$$

and

$$\begin{aligned}
(\Delta u\, \eta_k) * \phi_k &\xrightarrow{k\to\infty} \Delta u,\\
\|\langle \nabla u, \nabla \eta_k\rangle * \phi_k\|_{L^2(\mathbb{R}^n)} &\leq \frac{C}{k}\|\nabla u\|_{L^2(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0,\\
\|(u\Delta\eta_k) * \phi_k\|_{L^2(\mathbb{R}^n)} &\leq \frac{C}{k^2}\|u\|_{L^2(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.
\end{aligned}$$

This proves that $(u_k)_{k\in\mathbb{N}}$ has the required properties.

(ii) First of all note that if $\Delta u \in L^2(\Omega)$ then L^2–elliptic regularity implies that $u \in H^2(\Omega)$. If the sequence $(u_k)_k$ exists, then L^2-elliptic estimate implies

$$\lim_{k\to\infty} \|(u - u_k)\|_{H^2(\Omega)} \leq C \lim_{k\to\infty} \|\Delta(u - u_k)\|_{L^2(\Omega)} = 0,$$

which means that $(u_k)_k$ approximates u in $H^2(\Omega)$, and since the sequence is compactly supported, this would imply $u \in H^2_0(\Omega)$. But since Ω is bounded we have

$$H^2_0(\Omega) \subsetneq (H^1_0 \cap H^2)(\Omega),$$

so the sequence cannot exist for a general u.

(iii) We have seen in (ii) that if the required approximating sequence exists, then it must be $u \in H_0^2(\Omega)$; vice versa by definition for every element $v \in H_0^2(\Omega)$ there exits a sequence in $(v_k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ so that $v_k \to v$ in $H^2(\Omega)$, and so a fortiori so that $v_k \to v$ in $H^1(\Omega)$ and $\Delta v_k \to \Delta v$ in $L^2(\Omega)$.

So the set with the required property is precisely $H_0^2(\Omega)$. □

Exercise 11.5 Let $\Omega \subset \mathbb{R}^n$ be open. Let $a^{ij} \colon \Omega \to \mathbb{R}$ be measurable functions for every $i, j \in \{1, \ldots, n\}$. A differential operator L in non-divergence form

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x)\partial_{ij}^2 u,$$

is called *uniformly elliptic in* Ω, if there exists $\lambda > 0$ such that for almost every $x \in \Omega$ and every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2. \tag{1}$$

For $a^{ij} \in C^2(\overline{\Omega})$ and $c \in C^0(\overline{\Omega})$, we say that $u \in H_0^1(\Omega)$ is a *weak solution* of

$$-Lu + cu = f \quad \text{in } \Omega, \tag{2}$$

if for every $\varphi \in H_0^1(\Omega)$ there holds

$$\sum_{i,j=1}^{n} \int_\Omega a^{ij}\partial_j u\, \partial_i\varphi + \partial_i a^{ij}\, \partial_j u\, \varphi\, dx + \int_\Omega cu\varphi\, dx = \int_\Omega f\varphi\, dx. \tag{3}$$

(i) Prove that a classical solution $u \in C^2(\Omega) \cap H_0^1(\Omega)$ of $-Lu + cu = f$ is also a weak solution.

(ii) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $f \in L^2(\Omega)$. Let $a^{ij} \in C^2(\overline{\Omega})$ satisfy (1). Find a condition on $c \in C^0(\overline{\Omega})$ so that (2) admits a unique weak solution $u \in H_0^1(\Omega)$.

Solution. (i) Follows immediately with integration by parts.

(ii) We seek to apply the Lax-Milgram theorem.

Given $u, \varphi \in H_0^1(\Omega)$, define the (not necessarily symmetric) bilinear form

$$B\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$B(u,\varphi) = \sum_{i,j=1}^n \int_\Omega a^{ij}\partial_j u\,\partial_i\varphi + \partial_i a^{ij}\,\partial_j u\,\varphi\,dx + \int_\Omega cu\varphi\,dx.$$

Then, for a constant $C = C(a^{ij}, c, \Omega) > 0$, there holds

$$\begin{aligned}
&|B(u,\varphi)| \\
&\le \sum_{i,j=1}^n \int_\Omega |a^{ij}||\nabla u||\nabla\varphi| + |\nabla a^{ij}||\nabla u||\varphi| + |c||u||\varphi|\,dx \\
&\le \sum_{i,j=1}^n \Big(\|a^{ij}\|_{C^0}\|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^2} + \|\nabla a^{ij}\|_{C^0}\|\nabla u\|_{L^2}\|\varphi\|_{L^2} + \|c\|_{C^0}\|u\|_{L^2}\|\varphi\|_{L^2}\Big) \\
&\le C\|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^2},
\end{aligned}$$

where we applied Poincaré inequality to u and φ.

Now ellipticity implies

$$\int_\Omega \sum_{i,j=1}^n a^{ij}\partial_j u\,\partial_i u\,dx \ge \int_\Omega \lambda|\nabla u|^2\,dx = \lambda\|\nabla u\|^2_{L^2(\Omega)};$$

on the other hand, integrating by parts yields

$$\sum_{i,j=1}^n \int_\Omega \partial_i a^{ij}\,\partial_j u\,u\,dx = \frac{1}{2}\sum_{i,j=1}^n \int_\Omega \partial_i a^{ij}\,\partial_j(u^2)\,dx = -\frac{1}{2}\sum_{i,j=1}^n \int_\Omega \partial^2_{ij}a^{ij}\,u^2\,dx;$$

consequently, we have

$$B(u,u) \ge \lambda\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega \Big(c - \frac{1}{2}\sum_{i,j=1}^n \partial^2_{ij}a^{ij}\Big)u^2\,dx.$$

Hence if (for instance) there holds

$$c(x) - \frac{1}{2}\sum_{i,j=1}^n \partial^2_{ij}a^{ij}(x) \ge 0 \quad \text{in } \Omega,$$

we conclude

$$B(u,u) \ge \lambda\|\nabla u\|^2_{L^2(\Omega)}.$$

and the Lax-Milgram Lemma (Satz 4.3.3) applies and (by Korollar 4.3.1) we obtain a unique $u \in H^1_0(\Omega)$ such that

$$\forall\varphi \in H^1_0(\Omega): \quad B(u,\varphi) = \int_\Omega f\varphi\,dx.$$

$\square$

Exercise 11.6 Given $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$ prove that

$$\frac{\partial u}{\partial x_i} \in H^1_0(\mathbb{R}^n_+)$$

for every $i \in \{1, \dots, n-1\}$.

Solution. Given $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$ and $h \in \mathbb{R} \setminus \{0\}$, let $D_{h,i}u \colon \mathbb{R}^n_+ \to \mathbb{R}$ be given by

$$D_{h,i}u(x) = \frac{u(x + he_i) - u(x)}{h},$$

where $e_i = (0, \dots, 0, 1, 0 \dots, 0, 0) \in \mathbb{R}^n$ has the entry 1 at position $i \in \{1, \dots, n-1\}$.

The translation by he_i is an isometry of $H^1(\mathbb{R}^n_+)$ and maps $C^\infty_c(\mathbb{R}^n_+)$ into itself, so it maps its closure $H^1_0(\mathbb{R}^n_+)$ into itself. Therefore, $u \in H^1_0(\Omega)$ implies $D_{h,i}u \in H^1_0(\mathbb{R}^n_+)$.

According to Satz 8.3.1.iii) the assumption $u \in H^2(\mathbb{R}^n_+)$ implies

$$\exists C < \infty \quad \forall h \in \mathbb{R}^n \setminus \{0\} : \quad \|D_{h,i}u\|_{H^1} \leq C.$$

Hence, there exists a sequence $h_k \xrightarrow{k\to\infty} 0$ such that $D_{h_k,i}u$ converges weakly in $H^1(\mathbb{R}^n_+)$ to some $v \in H^1(\mathbb{R}^n_+)$ as $k \to \infty$. Since $H^1_0(\mathbb{R}^n_+)$ is a closed subspace of $H^1(\mathbb{R}^n_+)$, it is weakly closed. Therefore, $v \in H^1_0(\mathbb{R}^n_+)$. Moreover, for any $\varphi \in C^\infty_c(\mathbb{R}^n_+)$ there holds

$$\begin{aligned}
\int_{\mathbb{R}^n_+} v\varphi \, dx &= \lim_{k\to\infty} \int_{\mathbb{R}^n_+} \frac{u(x + h_k e_i) - u(x)}{h_k} \varphi(x) \, dx \\
&= \lim_{k\to\infty} \frac{1}{h_k} \Big(\int_{\mathbb{R}^n_+} u(x + h_k e_i)\varphi(x) \, dx - \int_{\mathbb{R}^n_+} u(x)\varphi(x) \, dx \Big) \\
&= \lim_{k\to\infty} \frac{1}{h_k} \Big(\int_{\mathbb{R}^n_+} u(y)\varphi(y - h_k e_i) \, dy - \int_{\mathbb{R}^n_+} u(x)\varphi(x) \, dx \Big) \\
&= -\lim_{k\to\infty} \int_{\mathbb{R}^n_+} u(x) \frac{\varphi(x) - \varphi(x - h_k e_i)}{h_k} \, dx \\
&= -\int_{\mathbb{R}^n_+} u \frac{\partial \varphi}{\partial x_i} \, dx.
\end{aligned}$$

By definition of weak derivative,

$$\frac{\partial u}{\partial x_i} = v \in H^1_0(\mathbb{R}^n_+)$$

and the claim follows. □

Hints to Exercises.

11.1 For (i), use the p–triangle inequality $|a+b|^p \leq C_p(|a|^p + |b|^p)$

For (ii) recall also Exercise 9.1.

11.2 For (i), construct the sequence explicitly;

For (ii), use the elliptic estimates.

11.5 Seek to apply the Lax-Milgram theorem.

11.6 Argue with the difference quotients.