Exercise 12.1 Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain, let $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be a weak solution of

$$-\Delta u = f \quad \text{in } \Omega,$$

Prove that then for every $\Omega' \subset\subset \Omega$, $\nabla u \in H^1(\Omega')$, and there holds

$$\|\nabla u\|_{H^1(\Omega')} \leq C\Big(\|f\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\Big), \tag{$\star$}$$

for some constant $C = C(\Omega, \Omega') > 0$.

Solution. The fact that $\nabla u \in H^1(\Omega')$ follows at once from the elliptic regularity theory (Satz 9.2.1).

To obtain the estimate, it suffices to note that

$$\nabla u = \nabla(u - \lambda) \quad \forall\, \lambda \in \mathbb{R},$$

so that by the interior elliptic estimate we have

$$\begin{aligned}
\|\nabla u\|_{H^1(\Omega')} &\leq \|\nabla(u-\lambda)\|_{H^1(\Omega')} \\
&\leq \|u-\lambda\|_{H^2(\Omega')} \\
&\leq C\Big(\|\Delta(u-\lambda)\|_{L^2(\Omega)} + \|u-\lambda\|_{L^2(\Omega)}\Big) \\
&= C\Big(\|f\|_{L^2(\Omega)} + \|u-\lambda\|_{L^2(\Omega)}\Big),
\end{aligned}$$

and so choosing $\lambda = ⨍_\Omega u\, dx$, Poincaré's inequality gives

$$\|u-\lambda\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)},$$

thus yielding ($\star$). □

Exercise 12.2 Let $B_R(x_0) \subset \mathbb{R}^n$ and let $u : B_R(x_0) \to \mathbb{R}$ be harmonic. Prove that the function

$$\varphi(r) = \frac{1}{r^n}\int_{B_r(x_0)} |u(x)|^2\, dx, \quad r \in (0, R),$$

is increasing.

Remark. This yields that for harmonic functions Satz 10.2.1 (i) holds with $C = 1$.

Solution. Without loss of generality we may assume $x_0 = 0$. In polar coordinates φ reads

$$\varphi(r) = \frac{1}{r^n} \int_0^r \int_{S^{n-1}} |u(\rho\vartheta)|^2 \, d\vartheta \rho^{n-1} \, d\rho,$$

so differentiating φ in r yields

$$\varphi'(r) = -\frac{n}{r^{n+1}} \int_0^r \int_{S^{n-1}} |u(\rho\vartheta)|^2 \, d\vartheta \rho^{n-1} \, d\rho + \frac{1}{r} \int_{S^{n-1}} |u(r\vartheta)|^2 \, d\vartheta.$$

We then have to prove that the right-hand side is non negative.

Since u is harmonic we have

$$\Delta |u|^2 = 2|\nabla u|^2 \geq 0,$$

so integrating this inequality and using the Divergence Theorem gives

$$0 \leq \int_{B_r} \Delta |u(x)|^2 \, dx = r^{n-1} \int_{S^{n-1}} \partial_r |u(r\vartheta)|^2 \, d\vartheta = r^{n-1} \frac{d}{dr} \int_{S^{n-1}} |u(r\vartheta)|^2 \, d\vartheta,$$

so the function $r \mapsto \int_{S^{n-1}} |u(r\vartheta)|^2 \, d\vartheta$ is non decreasing. Consequently

$$\begin{aligned}
&-\frac{n}{r^{n+1}} \int_0^r \int_{S^{n-1}} |u(\rho\vartheta)|^2 \, d\vartheta \rho^{n-1} \, d\rho \\
&\leq -\frac{n}{r^{n+1}} \int_0^r \rho^{n-1} \, d\rho \int_{S^{n-1}} |u(r\vartheta)|^2 \, d\vartheta \\
&= -\frac{1}{r} \int_{S^{n-1}} |u(r\vartheta)|^2 \, d\vartheta,
\end{aligned}$$

and this information is precisely what is needed to conclude that $\varphi'(r) \geq 0$. $\square$

Exercise 12.3 Let $\Omega \subset\subset \mathbb{R}^n$ be a bounded, regular domain, let $a^{ij} = a^{ji} \in C^1(\overline{\Omega})$ satisfy the uniform ellpticity condition

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall\, x \in \Omega,$$

for some $\lambda > 0$. Let $\alpha \in (0,1)$ be fixed and suppose you know that, for every $f \in C^{0,\alpha}(\overline{\Omega})$, the weak solution $u \in H_0^1(\Omega)$ to the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n \partial_i\Big(a^{ij}(x)\partial_j u\Big) = f & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\triangle$}$$

belongs to $C^{2,\alpha}(\overline{\Omega})$. Prove that then the L^2–elliptic estimate (Satz 9.5.1) is enough to deduce that there holds

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C\|f\|_{C^{0,\alpha}(\Omega)},$$

for a constant C that does not depend on u and f.

Solution. Let

$$\mathcal{L} : L^2(\Omega) \to H^2(\Omega)$$

be the linear map that to f associates the unique (weak) solution to ($\triangle$) which we know to be in $H^2(\Omega)$ by L^2–elliptic estimates (Satz 9.5.1).

If we restrict $\mathcal{L}$ to the subspace

$$X = C^{0,\alpha}(\overline{\Omega}) \subset L^2(\Omega),$$

then by assumption we get that $\mathcal{L}$ maps X into

$$Y = C^{2,\alpha}(\overline{\Omega}) \subset H^2(\Omega).$$

If we endow these spaces with the norms

$$\begin{aligned} \|\cdot\|_X &= \|\cdot\|_{C^{0,\alpha}(\Omega)}, \\ \|\cdot\|_Y &= \|\cdot\|_{C^{2,\alpha}(\Omega)}, \end{aligned}$$

and we are able to prove that $\mathcal{L}$ is continuous from X to Y, we are done.

The spaces in question are Banach and so by the Closed Graph Theorem it is enough to prove that, if $(f_k, \mathcal{L}(f_k))_k$ is a sequence in $X \times Y$ so that

$$(f_k, \mathcal{L}(f_k)) \xrightarrow{k\to\infty} (f, w) \quad \text{in } X \times Y,$$

then $w = \mathcal{L}(f)$. Now, by definition $u_k = \mathcal{L}(f_k)$ satisfies

$$\sum_{ij=1}^{n} \int_\Omega a^{ij}(x)\, \partial_j u_k\, \partial_i \varphi\, dx = \int_\Omega f_k\, \varphi\, dx, \quad \forall\, \varphi \in C_c^\infty(\Omega),$$

but since $f_k \to f$ in $C^0(\overline{\Omega})$ and $\nabla u_k \to \nabla w$ in $C^0(\overline{\Omega})$, we may pass to the limit in this expression and deduce that w is a weak solution of ($\triangle$), and hence that $\mathcal{L}(f) = w$. $\square$

Exercise 12.4 Let $\Omega \subset \mathbb{R}^n$ be a bounded, regular domain.

(i) Prove that for every $\alpha \geq 1$, every $u \in C^1(\overline{\Omega})$ and every vector field $T \in C^1(\overline{\Omega}, \mathbb{R}^n)$, there holds

$$\int_\Omega |u|^\alpha \operatorname{div}(T)\, dx + \alpha \int_\Omega |u|^{\alpha-1} \langle \nabla u, T\rangle\, dx = \int_{\partial\Omega} |u|^\alpha \langle T, \nu\rangle\, d\sigma,$$

where ν is the exerior unit normal field on $\partial\Omega$.

(ii) Argue suitably on T and α and u to deduce that, for every $1 \leq p \leq n$, the usual continuous embedding for the trace operator can be improved to be

$$\cdot|_{\partial\Omega} : W^{1,p}(\Omega) \to L^{p^\sharp}(\partial\Omega), \quad u \mapsto u|_{\partial\Omega},$$

where

$$p^\sharp = \frac{(n-1)p}{n-p} \quad \text{when } p < n,$$

or any number in $[1, \infty)$ when $p = n$.

Solution. (i) Immediate integrating by parts.

(ii) Let us assume first that $u \in C^1(\overline{\Omega})$ and $p < n$. Let T any be fixed extension in $C^1(\overline{\Omega}, \mathbb{R}^n)$ of the unit normal field ν of Ω, letting by Hölder's inequality we may estimate

$$\left| \int_\Omega |u|^{\alpha-1} \langle \nabla u, T\rangle\, dx \right| \leq C \|u\|^{\alpha-1}_{L^{p'(\alpha-1)}(\Omega)} \|\nabla u\|_{L^p(\Omega)},$$

and similarly

$$\left| \int_\Omega |u|^\alpha \operatorname{div}(T)\, dx \right| \leq C \|u\|^\alpha_{L^\alpha(\Omega)},$$

where p' is the conjugate of p. Now if $p^* = \frac{np}{n-p}$ denotes the Sobolev conjugate of p, we determine α so that

$$p'(\alpha - 1) \stackrel{!}{=} p^*$$

i.e. $\alpha = p^\sharp$ as above.

By (i) we deduce the estimate

$$\|u\|^{p^\sharp}_{L^{p^\sharp}(\partial\Omega)} \leq C\Big(\|\nabla u\|_{L^p(\Omega)} \|u\|^{p^\sharp - 1}_{L^{p^*}(\Omega)} + \|u\|^{p^\sharp}_{L^{p^\sharp}(\Omega)} \Big),$$

but since Ω is bounded and $p^\sharp < p^*$, we have $\|u\|_{L^{p^\sharp}(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)}$, whence

$$\|u\|_{L^{p^\sharp}(\partial\Omega)} \leq C\Big(\|\nabla u\|_{L^p(\Omega)}^{\frac{1}{p^\sharp}} \|u\|_{L^{p^*}(\Omega)}^{\frac{p^\sharp-1}{p^\sharp}} + \|u\|_{L^{p^*}(\Omega)}\Big);$$

using Young's inequality on the first term on the right hand side and then Sobolev inequality finally gives

$$\|u\|_{L^{p^\sharp}(\partial\Omega)} \leq C\Big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^{p^*}(\Omega)}\Big) \leq C\|u\|_{W^{1,p}(\Omega)}.$$

where $C = C(n, \Omega, p)$.

Now for general $u \in W^{1,p}(\Omega)$ $(p < n)$ we may argue, as usual, by approximation: if $(u_k)_k \in C^\infty(\overline{\Omega})$ converges to u, then the above inequality gives that $(u|_{\partial\Omega})_k$ is Cauchy in $L^{p^\sharp}(\partial\Omega)$, and thus by uniqueness of the limit that $u|_{\partial\Omega}$ has to be in $L^{p^\sharp}(\partial\Omega)$.

For $p = n$, since $W^{1,n}(\Omega) \hookrightarrow \bigcap_{1\leq p<n} W^{1,p}(\Omega)$, the thesis follows at once. $\square$

Exercise 12.5 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Given $A_0, A_1 \in L(X,Y)$ we define $A_t = (1-t)A_0 + tA_1$ for every $t \in [0,1]$ and assume that there exists $C > 0$ so that, for every $t \in [0,1]$ and every $x \in X$ there holds

$$\|x\|_X \leq C\|A_t x\|_Y. \qquad (*)$$

Prove that the statements

(i) A_0 is surjective,

(ii) A_1^* (the dual of A_1) is injective with closed range,

are equivalent by using the following method: define

$$I = \{t \in [0,1] \mid A_t \text{ is surjective}\}$$

and prove

(a) Either if (i) or (ii) hold, then $I \neq \emptyset$.

(b) The set $I \subset [0,1]$ is relatively open.

(c) The set $I \subset [0,1]$ is closed.

Combine (a), (b) and (c) to show $I = [0,1]$ and conclude.

Solution. We proceed following the outlined scheme.

(a) Let $I := \{t \in [0,1] \mid A_t \text{ is surjective}\}$. If we assume statement (i), then $0 \in I$. If we assume statement (ii), then $1 \in I$ by Satz 6.2.2. Therefore, $I \neq \emptyset$ in both cases.

(b) Let $t_0 \in I := \{t \in [0,1] \mid A_t \text{ is surjective}\}$. Assumption $(*)$ implies that A_{t_0} is also injective and that the inverse is continuous: $A_{t_0}^{-1} \in L(Y,X)$. For any $t \in [0,1]$, we have

$$\begin{aligned} A_t &= A_{t_0} - (A_{t_0} - A_t) = \Big(1 - (A_{t_0} - A_t)A_{t_0}^{-1}\Big)A_{t_0}, \\ A_{t_0} - A_t &= (1-t_0)A_0 + t_0A_1 - (1-t)A_0 - tA_1 = (t-t_0)(A_0 - A_1). \end{aligned}$$

Let $B := (A_{t_0} - A_t)A_{t_0}^{-1} \in L(Y,Y)$. By Satz 2.2.7 the operator $(1-B)$ is invertible with inverse $(1-B)^{-1} \in L(Y,Y)$ and in particular surjective, if $\|B\| < 1$. Since

$$\|B\| \leq \|A_{t_0} - A_t\|\|A_{t_0}^{-1}\| = |t - t_0|\|A_0 - A_1\|\|A_{t_0}^{-1}\|$$

we guarantee surjectivity of $(1-B)$ if $t \in [0,1]$ satisfies $|t - t_0| < \big(\|A_0 - A_1\|\|A_{t_0}^{-1}\|\big)^{-1}$. In this case we obtain that A_t is surjective, since A_{t_0} is surjective by assumption. Therefore, the set $I \subset [0,1]$ is open.

(c) Let $(t_k)_{k\in\mathbb{N}}$ be a sequence in I such that $t_k \to t_\infty$ as $k \to \infty$ for some $t_\infty \in [0,1]$. We claim that $A_{t_\infty} \in L(X,Y)$ is surjective. Let $y \in Y$ be arbitrary. Since $t_k \in I$, there exists $x_k \in X$ such that $A_{t_k}x_k = y$ for every $k \in \mathbb{N}$. Moreover, by assumption $(*)$,

$$\begin{aligned} \|x_k - x_n\|_X &\leq C\|A_{t_k}(x_k - x_n)\|_Y \\ &= C\|A_{t_k}x_k - A_{t_n}x_n + (A_{t_n} - A_{t_k})x_n\|_Y \\ &= C\|(A_{t_n} - A_{t_k})x_n\|_Y \\ &\leq C\|A_{t_n} - A_{t_k}\|\|x_n\|_X \\ &\leq C^2|t_k - t_n|\|A_0 - A_1\|\|A_{t_n}x_n\|_Y = C^2|t_k - t_n|\|A_0 - A_1\|\|y\|_Y \end{aligned}$$

which implies that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in X. Since $(X, \|\cdot\|_X)$ is complete, $(x_k)_{k\in\mathbb{N}}$ has a limit $x_\infty \in X$. Moreover,

$$\begin{aligned} \|y - A_{t_\infty}x_\infty\|_Y &= \|A_{t_k}x_k - A_{t_\infty}x_\infty\|_Y \\ &= \|(A_{t_k} - A_{t_\infty})x_k + A_{t_\infty}(x_k - x_\infty)\|_Y \\ &\leq C\|A_{t_k} - A_{t_\infty}\|\|y\|_Y + \|A_{t_\infty}\|\|x_k - x_\infty\|_X \\ &\leq C|t_\infty - t_k|\|A_0 - A_1\|\|y\|_Y + \|A_{t_\infty}\|\|x_k - x_\infty\|_X \xrightarrow{k\to\infty} 0. \end{aligned}$$

Hence, $A_{t_\infty} x_\infty = y$. Since $y \in Y$ is arbitrary, $t_\infty \in I$ follows. Therefore, the set $I \subset [0,1]$ is closed.

Since $[0,1]$ is a connected topological space and $I \subset [0,1]$ both open and closed by (b) and (c), we have either $I = \emptyset$ or $I = [0,1]$. According to Satz 6.2.2, A_1 is surjective if and only if A_1^* is injective with closed image. Hence, equivalence of (i) and (ii) follows:

$$\begin{array}{ccccccccc} \text{(i)} & \Leftrightarrow & 0 \in I & \Rightarrow & I = [0,1] & \Rightarrow & A_1 \text{ surjective} & \Leftrightarrow & \text{(ii)} \\ \text{(ii)} & \Leftrightarrow & 1 \in I & \Rightarrow & I = [0,1] & \Rightarrow & A_0 \text{ surjective} & \Leftrightarrow & \text{(i).} \end{array}$$

□

Hints to Exercises.

12.1 Use at your advantage that the desired estimate ($\star$) is invariant by additive constants.

12.2 Look at $\Delta|u|^2$: use the Divergence theorem to deduce a useful information from it.

12.3 Use suitably the Closed Graph Theorem.

12.5 Recall the basic fact about operator with closed image (§6.2).