Prof. Dr. Urs Lang

Exercise Sheet 1

1. Divergence Theorem

Let $M \subset \mathbb{R}^3$ be a compact 3-dimensional manifold with boundary, $N \colon \partial M \to S^2$ the outward pointing unit normal,

$$\pi = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$$

a 2-form on $\mathbb{R}^3$ and $X = (f, g, h)$.

(a) Show that $d\pi = \operatorname{div}(X) dx \wedge dy \wedge dz$.

(b) Deduce the Divergence Theorem

$$\int_M \operatorname{div}(X) \, \mathrm{dVol} = \int_{\partial M} \langle X, N \rangle \, \mathrm{d}A$$

from the Theorem of Stokes for differential forms.

2. De Rham Cohomology of T^2

Determine the de Rham cohomology of the torus T^2.

3. Tensor Fields

Let T be a $(1, 2)$-tensor field on M^m. Let (φ, U) and (ψ, U) be two charts on M. Show that the component ${}^{\psi}T^c_{ab}$ of T with respect to ψ depends on the components ${}^{\varphi}T^k_{ij}$ of T with respect to φ by the following relation:

$${}^{\psi}T^c_{ab} = \sum_{i,j,k=1}^{m} \frac{\partial \psi^c}{\partial \varphi^k} \frac{\partial \varphi^i}{\partial \psi^a} \frac{\partial \varphi^j}{\partial \psi^b} {}^{\varphi}T^k_{ij}.$$

Submission: until 12:15, March 3rd, in HG J68.