Exercise Sheet 10

1. Non-positive sectional curvature

Let M be a Hadamard manifold. Show that if $\gamma_1, \gamma_2 \colon \mathbb{R} \to M$ are two geodesics, then the function $s \mapsto d(\gamma_1(s), \gamma_2(s))$ is convex.

Hint: Use the second variation formula.

2. Some consequences of non-positive sectional curvature

Let M be a Hadamard manifold. Prove the following:

(a) For each $p \in M$, the map $(\exp_p)^{-1} \colon M \to TM_p$ is 1-Lipschitz.

(b) For $p, x, y \in M$, it holds

$$d(p,x)^2 + d(p,y)^2 - 2d(p,x)d(p,y)\cos\gamma \le d(x,y)^2,$$

where γ denotes the angle in p.

(c) Let m denote the midpoint of the geodesic xy in M and let $p \in M$. Then we have

$$d(p,m)^2 \le \frac{d(p,x)^2 + d(p,y)^2}{2} - \frac{1}{4}d(x,y)^2.$$

Hint: Note that in $\mathbb{R}^2$ this formula holds with an equality.

3. Isometries with bounded orbits.

Let M be a Hadamard manifold. Prove the following:

(a) If $Y \subset M$ is a bounded set, then there is a unique point $c_Y \in M$ such that $Y \subset \overline{B}(c_Y, r)$, where $r := \inf\{s > 0 : \exists x \in M \text{ such that } Y \subset \overline{B}(x,s)\}$. We call c_Y the *center* of Y.

(b) Let γ be an isometry of M. Then γ is elliptic if and only if M has a bounded orbit. Furthermore, if γ^n is elliptic for some integer $n \neq 0$, then γ is elliptic.

Submission: until 12:15, May 12th, by email at the following address:
`tommaso.goldhirsch@math.ethz.ch`