# Exercise Sheet 12

**1. Characterization of the cut value**

Prove Lemma 6.4 from the lecture:
Suppose that $M$ is complete. Let $c_u\colon \mathbb{R} \to M$, $c_u(t) := \exp_p(tu)$, be a unit speed geodesic. If the cut value $t_u$ is finite then (at least) one of the following holds for $t = t_u$:

(i) $c_u(t)$ is the first conjugate point of $p$ along $c_u|_{[0,t]}$,

(ii) there exists $v \in TM_p$, $|v| = 1$, $v \neq u$ with $c_u(t) = c_v(t)$.

Conversely, if (i) or (ii) is satisfied for some $t \in (0, \infty)$, then $t_u \leq t$.

**2. The cut locus**

Let $M$ be a complete manifold and let $p \in M$.

(i) Prove that the function $u \mapsto t_u$, defined on $S^{m-1} \subset TM_p$, is continuous.

(ii) Show that $M = \exp_p(U_p) \cup \mathrm{Cut}(p)$, where the union is disjoint. Moreover, we have a diffeomorphism $\exp_p|_{U_p}\colon U_p \to M \setminus \mathrm{Cut}(p)$.

**3. The injectivity radius of compact manifolds**

Let $M$ be a complete manifold and let $p \in M$.

(i) If $q \in \mathrm{Cut}(p)$ realizes the minimum distance from $p$ to $\mathrm{Cut}(p)$, then either there is a minimal geodesic $c_u$ from $p$ to $q$ along which $q$ is conjugate to $p$, or there are precisely two minimal geodesics $c_u, c_v$ from $p$ to $q$ such that $c_v'(t_v) = -c_u'(t_u)$.

(ii) If $M$ is a compact Riemannian manifold and $\kappa > 0$ is such that $\kappa \geq \sec$, then either $\mathrm{inj}(M) \geq D_\kappa$, or there exists a shortest closed geodesic $c$ in $M$ with $L(c) = 2\,\mathrm{inj}(M)$.

**Submission:** until 12:15, May 26th, by email at the following address:
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