Exercise Sheet 2

1. Left-invariant Vector Fields

Let G be a Lie group. Show that

(a) Left-invariant vector fields on G are smooth.

(b) If X, Y are left-invariant vector fields on G, then so is $[X, Y]$.

(c) If $F\colon H \to G$ is a Lie group homomorphism or isomorphism, then the differential $dF_e\colon TH_e \to TG_e$ is a Lie algebra homomorphism or isomorphism, respectively.

2. Unit Quaternions

(a) Show that the Lie group $S^3 \subset \mathbb{H}$ of unit quaternions is isomorphic to $\mathrm{SU}(2)$.

Hint: Consider the map

$$a + bi + cj + dk \longmapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

(b) Show that $S^3, \mathrm{SU}(2)$ and $\mathrm{SO}(3)$ have isomorphic Lie algebras.

3. Exponential Map

Let $0 < \alpha \neq 1$. Show that there doesn't exist $A \in \mathbb{R}^{2\times 2}$ with

$$e^A = \begin{pmatrix} -\alpha & 0 \\ 0 & -\frac{1}{\alpha} \end{pmatrix}$$

and conclude that the exponential map of a connected Lie group is not necessarily surjective (for example, consider the Lie group $\mathrm{GL}^+(n, \mathbb{R})$).

Submission: until 12:15, March 10th, in HG J68.