Exercise Sheet 3

1. Bi-invariant Metrics

A Riemannian metric $\langle\cdot,\cdot\rangle$ on a Lie group G is called *bi-invariant* if for all $g \in G$ the left-translation $l_g\colon G \to G$, $l_g(x) := gx$, and the right-translation $r_g\colon G \to G$, $r_g(x) := xg$, are isometries.

a) Show that for $G = \mathrm{SO}(n,\mathbb{R})$, $TG_g = \{(g, gA) :\ A \in \mathbb{R}^{n\times n}, A = -A^{\mathrm{T}}\}$,

$$\langle (g, gA), (g, gB)\rangle := \frac{1}{2}\mathrm{tr}(AB^{\mathrm{T}})$$

defines a bi-invariant metric on G.

b) Show that every compact Lie group admits a bi-invariant metric.

Hint: Define first a left-invariant metric on G, then use an appropriate integration over G.

c) Let G be a Lie group with a bi-invariant metric and let D be the corresponding Levi-Civita connection. Prove that for left-invariant vector fields $X, Y \in \Gamma(TG)$ we have

$$D_X Y = \frac{1}{2}\,[X, Y]\,.$$

2. The Levi-Civita connection on a submanifold

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with Levi-Civita connection $\bar{D}$, and let M be a submanifold of $\bar{M}$, equipped with the induced metric $g := i^*\bar{g}$, where $i\colon M \to \bar{M}$ is the inclusion map.

Show that the Levi-Civita connection D of (M, g) satisfies $D_X Y = (\bar{D}_X Y)^{\mathrm{T}}$ for all $X, Y \in \Gamma(TM)$, where the superscript T denotes the component tangential to M and $\bar{D}_X Y$ is defined(!) as $\bar{D}_X Y := \bar{D}_{\bar{X}}\bar{Y}$ for any extensions $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$ of X, Y.

3. Gradient and Hessian form

Let (M, g) be a Riemannian manifold, D the Levi-Civita connection and $f\colon M \to \mathbb{R}$ a smooth function on M.

a) The *gradient* $\mathrm{grad} f \in \Gamma(TM)$ is defined by

$$df(X) = g(\mathrm{grad} f, X), \quad \forall X \in \Gamma(TM).$$

Compute $\mathrm{grad} f$ in local coordinates.

b) The *Hessian form* $\mathrm{Hess}(f) \in \Gamma(T_{0,2}M)$ is defined by

$$\mathrm{Hess}(f)(X, Y) = g(D_X \mathrm{grad} f, Y), \quad \forall X, Y \in \Gamma(TM).$$

Prove that $\mathrm{Hess}(f)$ is symmetric and compute $\mathrm{Hess}(f)$ in local coordinates.

Submission: until 12:15, March 17th, in HG J68.