Exercise Sheet 5

1. Constant sectional curvature

Let (M, g) be a Riemannian manifold with constant sectional curvature $\sec(E) = \kappa \in \mathbb{R}$ for all $E \in G_2(TM)$. Show that

$$R(X, Y)W = \kappa \left(g(Y, W)X - g(X, W)Y \right).$$

2. Ricci curvature

Let (M, g) be a 3-dimensional Riemannian manifold. Show the following:

a) The Ricci curvature ric uniquely determines the Riemannian curvature tensor R.

b) If M is an Einstein manifold, that is, a Riemannian manifold (M, g) with $\mathrm{ric} = kg$ for some $k \in \mathbb{R}$, then the sectional curvature sec is constant.

3. Divergence and Laplacian

Let (M, g) be a Riemannian manifold with Levi-Civita connection D. The *divergence* $\mathrm{div}(Y)$ of a vector field $Y \in \Gamma(TM)$ is the contraction of the $(1,1)$-tensor field $DY : X \mapsto D_X Y$ and the *Laplacian* $\Delta \colon C^\infty(M) \to C^\infty(M)$ is defined by $\Delta f := \mathrm{div}(\mathrm{grad}\, f)$. Show that:

a) $\mathrm{div}(fY) = Y(f) + f \,\mathrm{div}\, Y$,

b) $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \mathrm{grad}\, f, \mathrm{grad}\, g\rangle$,

c) Compute Δf in local coordinates.

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