Exercise Sheet 9

1. Almost complex structure

An almost complex structure J on a manifold M^m is a $(1,1)$-tensor field with the following property: if for every $p \in M$ we denote by $J_p \colon TM_p \to TM_p$ the linear map associated with J (recall Theorem T.3), then

$$J_p \circ J_p = -\operatorname{id}_{TM_p}.$$

Prove that every complex manifold admist an almost complex structure.
Hint: Composed with the differential of a complex chart $\varphi \colon U \to \varphi(U) \subset \mathbb{C}^n$, J_p amounts to the multiplication by i.

2. Kähler manifolds

Let M be a complex manifold with an almost complex structure $J \in \Gamma(T_{1,1}M)$ (as in Exercise 1). Suppose that M is endowed with an hermitian metric, that is, $g_p(J_pv, J_pw) = g_p(v, w)$ for all $p \in M$ and $v, w \in TM_p$. Show that

$$\omega(X, Y) \coloneqq g(X, JY) \quad (X, Y \in \Gamma(TM))$$

defines a 2-form $\omega \in \Omega^2(M)$, which is closed if and only if J is parallel (i.e. $DJ = D^{1,1}J \equiv 0$).

3. Translations

Suppose that Γ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

a) Show that there exist linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \left\{ x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \dots, z_k) \in \mathbb{Z}^k \right\} \simeq \mathbb{Z}^k.$$

b) Let l denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that l equals the length of the shortest non-zero vector of the form $\sum_{i=1}^{k} z_i v_i$ with $z_i \in \mathbb{Z}$ as above.

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