Solutions 1

1. Divergence Theorem

Let $M \subset \mathbb{R}^3$ be a compact 3-dimensional manifold with boundary, $N\colon \partial M \to S^2$ the outward pointing unit normal,

$$\pi = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$$

a 2-form on $\mathbb{R}^3$ and $X = (f, g, h)$.

(a) Show that $d\pi = \operatorname{div}(X) dx \wedge dy \wedge dz$.

(b) Deduce the Divergence Theorem

$$\int_M \operatorname{div}(X)\, \mathrm{dVol} = \int_{\partial M} \langle X, N \rangle\, \mathrm{d}A$$

from the Theorem of Stokes for differential forms.

Solution.
(a)

$$\begin{aligned} d\pi &= \frac{\partial f}{\partial x} dx \wedge dy \wedge dz + \frac{\partial g}{\partial y} dy \wedge dz \wedge dx + \frac{\partial h}{\partial z} dz \wedge dx \wedge dy \\ &= \big(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\big) dx \wedge dy \wedge dz \\ &= \operatorname{div}(X) dx \wedge dy \wedge dz. \end{aligned}$$

(b) Let (U, φ) be a chart of ∂M such that for (the parametrization) $\psi := \varphi^{-1}\colon \varphi(U) \subset \mathbb{R}^2 \to \partial M$ the outward pointing unit normal is given by

$$N \circ \psi = \frac{\partial_x \psi \times \partial_y \psi}{\|\partial_x \psi \times \partial_y \psi\|} = \frac{\psi_1 \times \psi_2}{\|\psi_1 \times \psi_2\|} = \frac{d\psi(e_1) \times d\psi(e_2)}{\|d\psi(e_1) \times d\psi(e_2)\|}.$$

Then

$$\int_U \pi = \int_{\varphi(U)} \psi^* \omega.$$

In order to integrate $\psi^*\omega$ we need to write it in the form $\psi^*\omega = a dx \wedge dy$ and find the coeffiecient function $a\colon \varphi(U) \to \mathbb{R}$.

Recall that for $q = \varphi(p)$, $a(q) = (a dx \wedge dy)_q(e_1, e_2)$ thus

$$\begin{aligned}
a(q) &= (\psi^*\pi)_q(e_1, e_2) \\
&= \pi_p(d\psi_q(e_1), d\psi_q(e_2)) \\
&= \pi_p(\psi_1(q), \psi_2(q)) \\
&= f(p) \underbrace{\det \begin{pmatrix} \psi_1^2 & \psi_2^2 \\ \psi_1^3 & \psi_2^3 \end{pmatrix} (q)}_{=\left(\psi_1(q)\times\psi_2(q)\right)^1} + g(p) \underbrace{\det \begin{pmatrix} \psi_1^3 & \psi_2^3 \\ \psi_1^1 & \psi_2^1 \end{pmatrix} (q)}_{\left(\psi_1(q)\times\psi_2(q)\right)^2} + h(p) \underbrace{\det \begin{pmatrix} \psi_1^1 & \psi_2^1 \\ \psi_1^2 & \psi_2^2 \end{pmatrix} (q)}_{\left(\psi_1(q)\times\psi_2(q)\right)^3} \\
&= \langle X(p), \psi_1(q) \times \psi_2(q) \rangle \\
&= \langle X \circ \psi(q), \frac{\psi_1(q) \times \psi_2(q)}{\|\psi_1(q) \times \psi_2(q)\|} \rangle \cdot \|\psi_1(q) \times \psi_2(q)\| \\
&= \langle X, N \rangle \circ \psi(q) \cdot \|\psi_1(q) \times \psi_2(q)\|.
\end{aligned}$$

It's a computation to see that $\|\psi_1(q) \times \psi_2(q)\| = \sqrt{\det \langle \psi_i(q), \psi_j(q) \rangle_{i,j=1,2}}$, so

$$\int_U \pi = \int_{\varphi(U)} \psi^*\omega = \int_{\varphi(U)} \langle X, N \rangle \circ \psi \cdot \|\psi_1(q) \times \psi_2(q)\| \, \mathrm{d}x \, \mathrm{d}y = \int_U \langle X, N \rangle \, \mathrm{d}A.$$

Using (a) and the theorem of Stokes (together with a partition of unity) we obtain

$$\int_M \operatorname{div} X \, \mathrm{dVol} = \int_M \operatorname{div}(X) dx \wedge dy \wedge dz = \int_M d\pi = \int_{\partial M} \pi = \int_{\partial M} \langle X, N \rangle \, \mathrm{d}A.$$

2. De Rham Cohomology of T^2

Determine the de Rham cohomology of the torus T^2.

Solution. Cover the torus T^2 using two (sets homeomorphic to) cylinders U, V so that they overlap slightly at the extremities. Then U and V are both homotopy equivalent to S^1, so

$$H^s_{dR}(U) \cong H^s_{dR}(V) \cong H^s_{dR}(S^1) \cong \begin{cases} \mathbb{R} & s = 0, 1, \\ 0 & s \geq 2; \end{cases}$$

the intersection $U \cap V$ is homotopy equivalent to the disjoint union of two copies of S^1 and it's not difficult to see (e.g by using the Mayer-Vietoris sequence) that

$$H^s_{dR}(U \cap V) \cong H^s_{dR}(S^1 \sqcup S^1) \cong H^s_{dR}(S^1) \oplus H^s_{dR}(S^1) \cong \begin{cases} \mathbb{R}^2 & s = 0, 1, \\ 0 & s \geq 2. \end{cases}$$

Using that $H^0_{dR}(T^2) \cong \mathbb{R}$ and, as seen in the lecture, $H^2_{dR}(T^2) \cong \mathbb{R}$, the Mayer-Vietoris sequence for this open cover is

$$0 \to H^0_{dR}(T^2) \to H^0_{dR}(U) \oplus H^0_{dR}(V) \to H^0_{dR}(U \cap V) \to H^1_{dR}(T^2)$$
$$\to H^1_{dR}(U) \oplus H^1_{dR}(V) \to H^1_{dR}(U \cap V) \to H^2_{dR}(T^2) \to 0,$$

which amounts to

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to H^1_{dR}(T^2) \to \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R} \to 0.$$

Since the alternating sum of the dimensions is 0, we conclude that $-1 + 2 - 2 + \dim H^1_{dR}(T^2) - 1 + 2 - 1 = 0$, and therefore $H^1_{dR}(T^2) \cong \mathbb{R}^2$.

3. Tensor Fields

Let T be a $(1,2)$-tensor field on M^m. Let (φ, U) and (ψ, U) be two charts on M. Show that the component ${}^{\psi}T^c_{ab}$ of T with respect to ψ depends on the components ${}^{\varphi}T^k_{ij}$ of T with respect to φ by the following relation:

$${}^{\psi}T^c_{ab} = \sum_{i,j,k=1}^{m} \frac{\partial \psi^c}{\partial \varphi^k} \frac{\partial \varphi^i}{\partial \psi^a} \frac{\partial \varphi^j}{\partial \psi^b} {}^{\varphi}T^k_{ij}.$$

Solution. For $a, b, c \in \{1, \dots, m\}$ we have

$${}^{\psi}T^c_{ab} = T(d\psi^c \otimes \frac{\partial}{\partial \psi^a} \otimes \frac{\partial}{\partial \psi^b})$$

so by writing T with respect to φ we see that

$$\begin{aligned}
{}^{\psi}T^c_{ab} &= \sum_{i,j,k=1}^{m} {}^{\varphi}T^k_{i,j} \frac{\partial}{\partial \varphi^k} \otimes d\varphi^i \otimes d\varphi^j \Big(d\psi^c \otimes \frac{\partial}{\partial \psi^a} \otimes \frac{\partial}{\partial \psi^b} \Big) \\
&= \sum_{i,j,k=1}^{m} {}^{\varphi}T^k_{i,j} \big(\frac{\partial}{\partial \varphi^k}(d\psi^c) \big) \cdot \big(d\varphi^i(\frac{\partial}{\partial \psi^a}) \big) \cdot \big(d\varphi^j(\frac{\partial}{\partial \psi^b}) \big) \\
&= \sum_{i,j,k=1}^{m} {}^{\varphi}T^k_{i,j} \big(d\psi^c(\frac{\partial}{\partial \varphi^k}) \big) \cdot \big(d\varphi^i(\frac{\partial}{\partial \psi^a}) \big) \cdot \big(d\varphi^j(\frac{\partial}{\partial \psi^b}) \big) \\
&= \sum_{i,j,k=1}^{m} {}^{\varphi}T^k_{i,j} \frac{\partial \psi^c}{\partial \varphi^k} \cdot \frac{\partial \varphi^i}{\partial \psi^a} \cdot \frac{\partial \varphi^j}{\partial \psi^b}.
\end{aligned}$$