Solutions 11

1. Projection in Hadamard manifolds

Let C be a closed convex subset of the Hadamard manifold M. Prove the following:

(a) For every point $p \in M$ there is a unique point $\pi(p) \in C$ such that $d(p, \pi(p)) = d(p, C)$. Moreover, if $q \in p\pi(p)$ then $\pi(q) = \pi(p)$.

(b) For $p \in M \setminus C$ and $y \in C$ with $y \neq \pi(p)$, we have $\measuredangle_{\pi(p)}(p, y) \geq \frac{\pi}{2}$.

(c) The projection map $\pi\colon M \to C$ is 1-Lipschitz.

Solutions. (a) Let $(p_n)_n \subset C$ be a sequence such that $d(p, p_n) \leq d(p, C) + \frac{1}{n}$. For $n, n' \in \mathbb{N}$, let $m_{n,n'} \in C$ be the midpoint of $p_n p_{n'}$. By exercise 2.c) of Serie 10, we get that

$$\begin{aligned}
\tfrac{1}{4} d(p_n, p_m)^2 &\leq \frac{d(p, p_n)^2 + d(p, p_{n'})^2}{2} - d(p, m_{n,n'})^2 \\
&\leq \frac{\left(d(p, C) + \frac{1}{n}\right)^2 + \left(d(p, C) + \frac{1}{n'}\right)^2}{2} - d(p, C)^2 \\
&= d(p, C) \left(\tfrac{1}{n} + \tfrac{1}{n'}\right) + \tfrac{1}{2}\left(\tfrac{1}{n^2} + \tfrac{1}{n'^2}\right).
\end{aligned}$$

Hence $(p_n)_n$ is Cauchy and converges to some $x \in C$.

If there is another $y \in C$ with $d(p, y) = d(p, C)$, we get for the midpoint $m \in C$ of xy

$$d(p, m)^2 \leq \frac{d(p, x)^2 + d(p, y)^2}{2} - \frac{1}{4} d(x, y)^2 < d(p, C)^2,$$

a contradiction. Hence uniqueness follows.

If $q \in p\pi(p)$, we have

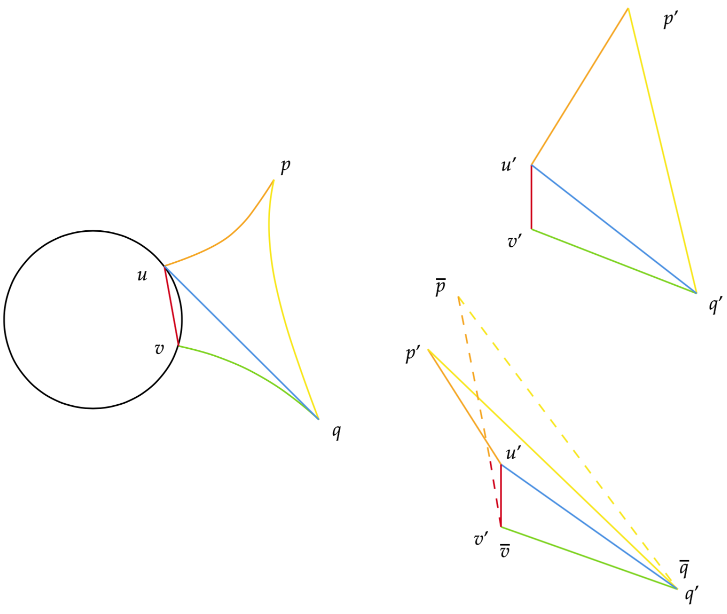
$$d(p, \pi(q)) \leq d(p, q) + d(q, \pi(q)) \leq d(p, q) + d(q, \pi(p)) = d(p, \pi(p))$$

and therefore $\pi(q) = \pi(p)$ by uniqueness of $\pi(p)$.

(b) If $\measuredangle_{\pi(p)}(p, y) < \frac{\pi}{2}$, there are $y' \in \pi(p)y$ and $p' \in \pi(p)p$ such that $\measuredangle_{\overline{\pi}(p)}(\overline{p}', \overline{y}') < \frac{\pi}{2}$ in the comparison triangle $(\overline{\pi}(p), \overline{p}', \overline{y}')$ of $(\pi(p), p', y')$. But then there is some $\overline{x} \in \overline{\pi}(p)\overline{y}'$ with $|\overline{x}\overline{p}'| < |\overline{\pi}(p)\overline{p}'|$. Let $x \in \pi(p)y'$ with $|\pi(p)x| = |\overline{\pi}(p)\overline{x}|$. Then, by the distance comparison (D^0), we get

$$|xp'| \leq |\overline{x}\overline{p}'| < |\overline{\pi}(p)\overline{p}'| = |\pi(p)p'|,$$

but $\pi(p') = \pi(p)$.



(c) Consider $p \neq q \notin C$ with $u := \pi(p)$, $v := \pi(q)$, $u \neq v$.

We construct a quadrilateral p, q, u', v' in $\mathbb{R}^2$ such that (p', u', q') is a comparison triangle for (p, u, q), (u', v', q') is a comparison triangle for (u, v, q) and the two triangles are joined along $u'q'$, which separates p' and v' in the plane. Note that by (b) and the angle comparison property (A^0) we know that

$$\measuredangle_{v'}(u', q') \geq \measuredangle_v(u, q) \geq \frac{\pi}{2}. \tag{1}$$

We subdivide the proof in two cases.
Case 1: $\measuredangle_{u'}(p', q') + \measuredangle_{u'}(q', v') \leq \pi$, then

$$\begin{aligned} \measuredangle_{u'}(p', v') &= \measuredangle_{u'}(p', q') + \measuredangle_{u'}(q', v') \\ &\geq \measuredangle_u(p, q) + \measuredangle_u(q, v) \\ &\geq \measuredangle_u(p, v) \\ &\geq \frac{\pi}{2}, \end{aligned}$$

where the first inequality follows by (A^0), the second is the triangle inequality for Alexandrov angles and the third follows from (b). Thus together with (1) we have $\measuredangle_{u'}(p', v') \geq \pi/2$ and $\measuredangle_{v'}(q', u') \geq \pi/2$ and therefore

$$|uv| = |u'v'| \leq |p'q'| = |pq|.$$

Case 2: $\measuredangle_{u'}(p', q') + \measuredangle_{u'}(q', v') \geq \pi$. In this case, consider the triangle $(\overline{v}, \overline{p}, \overline{q})$ in $\mathbb{R}^2$, with $|\overline{v}\overline{q}| = |v'q'|$, $|\overline{p}\overline{q}| = |p'q'|$ and $|\overline{v}\overline{p}| = |v'u'| + |u'p'|$. By Alexandrov's

Lemma and (1), $\measuredangle_{\overline{v}}(\overline{p}, \overline{q}) \geq \measuredangle_{v'}(u', q') \geq \pi/2$, thus $|\overline{vp}| \leq |\overline{pq}|$, so

$$|v'u'| \leq |v'u'| + |u'p'| = |\overline{vp}| \leq |\overline{pq}| = |p'q'|,$$

which implies $|uv| \leq |pq|$.

2. Asymptotic expansion of the circumference

Let M be a manifold, $E \subset TM_p$ a linear 2-plane and $\gamma_r \subset E$ a circle with center 0 and radius $r > 0$ sufficiently small. Show that

$$L(\exp(\gamma_r)) = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right)$$

for $r \to 0$.

Solutions. Let $v, w \in TM_p$ be an orthonormal basis of E. Then the circle can be parametrized by $\gamma_r(\varphi) = r(v \cos\varphi + w \sin\varphi)$. For some fixed $\varphi_0 \in [0, 2\pi]$, consider the Jacobi field $Y_{\varphi_0}(r)$ associated to the geodesic variation $V(\varphi, r) := \exp(\gamma_r(\varphi))$ of the geodesic $c_{\varphi_0}(r) := \exp(\gamma_r(\varphi_0))$. Then it holds

$$L(\exp(\gamma_r)) = \int_0^{2\pi} |Y_\varphi(r)| \, d\varphi.$$

We will now compute the Taylor expansion for $|Y_0(r)|$ (compare with Serie 7, Exercise 3), all other cases are similar. We have $Y_0(0) = 0$ and $Y_0'(0) = w$. From the Jacobi equation we also get

$$Y_0''(0) = -R\left(Y_0, c_0'\right) c_0' \Big|_{r=0} = 0.$$

Now taking the derivative of the Jacobi equation, we get

$$Y_0'''(0) = -\frac{D}{dr} R\left(Y_0, c_0'\right) c_0' \Big|_{r=0} = -R\left(Y_0', c_0'\right) c_0' \Big|_{r=0} = -R(w, v)v.$$

It follows that

$$|Y_0(r)| = r - \frac{R(w, v, w, v)}{6} r^3 + \mathcal{O}(r^4).$$

Therefore, we finally get

$$L(\exp(\gamma_r)) = \int_0^{2\pi} \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right) d\varphi = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right),$$

as it was to show.

3. Alexandrov curvature and sectional curvature

(a) Determine the circumference of a circle in the model space $\mathbb{M}^2_\kappa$ and compute its Taylor expansion.

(b) Prove that if a connected Riemannian manifold M has curvature $\geq \kappa$ in the sense of Alexandrov, then it has bounded sectional curvature $\sec \geq \kappa$.

Solutions. (a) We start with the case $\kappa < 0$. First, consider a horizontal circle C_r of radius r in $\mathbb{H}^2$, seen as a subspace of the Minkowski space $\mathbb{R}^{2,1}$. Recall that a geodesic starting at e_3 and going into the direction e_1 which is parametrized by arclength is given by $c(t) = e_3 \cosh t + e_1 \sinh t$. Hence, the Euclidean radius of C_r is $\sinh r$ and thus $L(C_r) = 2\pi \sinh r$. Now, as $M^2_\kappa = \frac{1}{\sqrt{-\kappa}} \mathbb{H}^2$ (as a length space[1]), we get

$$L(C_r) = 2\pi \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} r).$$

Of course we have $L(C_r) = 2\pi r$ for $\kappa = 0$.

For $\kappa > 0$, observe again that a circle in S^2 of radius r has Euclidean radius $\sin r$. From the scaling $M^2_\kappa = \frac{1}{\sqrt{\kappa}} S^2$ it follows that

$$L(C_r) = 2\pi \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} r).$$

In all three cases, we therefore get that

$$L(C_r) = 2\pi \operatorname{sn}_\kappa(r) = 2\pi \left(r - \frac{\kappa}{6} r^3 + \mathcal{O}(r^5) \right).$$

(b) We prove the statement in the case curvature $\geq \kappa$. For $\leq \kappa$, the same argument works with reversed inequalities.

As for a curve $c\colon I \to M$ we have $L(c) = \inf \sum_{i=1}^n d(c(t_{i-1}), c(t_i))$ where $t_i \in I$ with $t_0 < \ldots < t_m$, it follows directly from the hinge comparison (H_κ) that $L(\exp_p(\gamma_r)) \leq L(C_r)$. Combining this with Exercise 2 gives

$$0 \leq L(C_r) - L(\exp_p(\gamma_r)) = 2\pi \left(\frac{1}{6} (\sec(E) - \kappa) r^3 + \mathcal{O}(r^4) \right)$$

for $r > 0$ small, and therefore $\sec(E) \geq \kappa$.

[1] That is, the length $L_\kappa(c)$ of a curve c in M^2_κ is satisfies $L_\kappa(c) = \frac{1}{\sqrt{-\kappa}} L_{\mathbb{H}}(c)$.