Solutions 2

1. Left-invariant Vector Fields

Let G be a Lie group. Show that

(a) Left-invariant vector fields on G are smooth.

(b) If X, Y are left-invariant vector fields on G, then so is $[X, Y]$.

(c) If $F\colon H \to G$ is a Lie group homomorphism or isomorphism, then the differential $dF_e\colon TH_e \to TG_e$ is a Lie algebra homomorphism or isomorphism, respectively.

Solution. (a) First of all note that for a smooth manifold M, a vector field $X\colon M \to TM$ is smooth if (and only if) for every open set U and every function $f \in \mathbb{C}^\infty(U)$, the function $Xf\colon U \to \mathbb{R}$ is smooth.

Now, let X be a left-invariant vector field on G, let $U \subset G$ be open and $f \in C^\infty(U)$. Choose a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to G$ with $\gamma(0) = e$ and $\gamma'(0) = X_e = X(e)$, then

$$\begin{aligned} Xf(p) &= X_p(f) = L_{p*}(X_e)(f) = d(L_p)_e(X_e)(f) = df_p d(L_p)_e(X_e) \\ &= d(f \circ L_p)_e(X_e) = \frac{d}{dt}\Big|_{t=0} f \circ L_p \circ \gamma(t) = \frac{d}{dt}\Big|_{t=0} f \circ \mu\big(p, \gamma(t)\big), \end{aligned}$$

where $\mu\colon G \times G \to G$ denotes the smooth multiplication map on G. The last expression is smooth in p and therefore Xf is smooth.

(b) A vector field X on G is left-invariant if and only if it is L_g-related to itself for all $g \in G$. Thus (b) follows from Exercise 1 (b) in Exercise Sheet 13 of Differential Geometry I (HS19).

(c) Suppose that $F\colon H \to G$ is a Lie group homomorphism. Let $X_e, Y_e \in \mathfrak{h}$, denote by $X, Y \in \Gamma(TH)$ the unique left-invariant vector fields associated with X_e and Y_e, respectively. Denote by $X', Y' \in \Gamma(TG)$ the unique left-invariant vector fields with $X'_e = dF_e(X_e) \in \mathfrak{g}$ and $Y'_e = dF_e(Y_e) \in \mathfrak{g}$, respectively.

We claim that X' is F-related to X, and similarly Y' is F-related to Y. Indeed, first notice that for $g, h \in H$

$$F \circ L_h(g) = F(hg) = F(h)F(g) = L_{F(h)} \circ F(g),$$

i.e. $F \circ L_h = L_{F(h)} \circ F$ (note that the two left-multiplication maps appearing in this identity are defined on different Lie groups). Then for $h \in H$

$$\begin{aligned} X'_{F(h)} &= d(L_{F(h)})_e(X'_e) = d(L_{F(h)})_e dF_e(X_e) = d(L_{F(h)} \circ F)_e(X_e) \\ &= d(F \circ L_h)_e(X_e) = dF_h d(L_h)_e(X_e) = dF_h X_h. \end{aligned}$$

Therefore by Exercise 1 (b) in Exercise Sheet 13 of Differential Geometry I (HS19) it follows that

$$[dF_e(X_e), dF_e(Y_e)] = [X'_e, Y'_e] = [X', Y']_e = dF_e([X,Y]_e) = dF_e([X_e, Y_e]).$$

This shows that $dF_e\colon \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism.

If F is an isomorphism then $d(F^{-1})_e$ is a Lie algebra homomorphism and $d(F^{-1})_e \circ dF$ is the identity.

2. Unit Quaternions

(a) Show that the Lie group $S^3 \subset \mathbb{H}$ of unit quaternions is isomorphic to $\mathrm{SU}(2)$.

Hint: Consider the map

$$a + bi + cj + dk \longmapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

(b) Show that $S^3, \mathrm{SU}(2)$ and $\mathrm{SO}(3)$ have isomorphic Lie algebras.

Solution. (a). Consider the map $\varphi\colon S^3 \subset \mathbb{H} \to \mathrm{SU}(2)$ defined by

$$a + bi + cj + dk \longmapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

It is well defined since

$$\det \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = 1$$

and

$$\begin{aligned} \varphi(a+bi + cj + dk) \cdot \overline{\varphi(a + bi + cj + dk)}^{\perp} \\ = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \cdot \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

because $a^2 + b^2 + c^2 + d^2 = 1$. We check that φ is a homomorphism. Let $a + bi + cj + dk, a' + b'i + c'j + d'k \in \mathbb{H}$, then

$$\begin{aligned} (a + bi + cj + dk) \cdot (a' + b'i + c'j + d'k) \\ &= (aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')i \\ &\quad + (ac' + ca' + db' - bd')j + (ad' + da' + bc' - cd')k \\ &=: A + Bi + Cj + Dk \end{aligned}$$

and one can show that

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \begin{pmatrix} a'+b'i & c'+d'i \\ -c'+d'i & a'-b'i \end{pmatrix} = \begin{pmatrix} A+Bi & C+Di \\ -C+Di & A-Bi \end{pmatrix}.$$

We now check surjectivity. Let $A \in \mathrm{SU}(2)$. From the conditions $A\overline{A}^{\perp} = \overline{A}^{\perp}A = \mathbb{1}$ and $\det A = 1$ we see that

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha}, \end{pmatrix}$$

with $\alpha\overline{\alpha}+\beta\overline{\beta} = 1$, for some $\alpha, \beta \in \mathbb{C}$. Writing $\alpha = a+bi$ and $\beta = c+di$ we see that $\varphi(a+bi+cj+dk) = A$. Injectivity follows because $\varphi(a+bi+cj+dk) = \mathbb{1}$ if and only if $a = 1$, $b = c = d = 0$ and therefore φ is a smooth group homomorphism with smooth inverse (which is also a homomorphism).

(b) It follows from part (a) and Exercise 1 (b) that S^3 and $\mathrm{SU}(2)$ have isomorphic Lie algebras. We now show that $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$have isomorphic Lie algebras. Recall from the lecture that

$$\begin{aligned} \mathfrak{so}(3) &= \Big\{ A \in \mathbb{R}^{3\times 3} \ : \ A = -A^{\perp} \Big\}, \\ \mathfrak{su}(2) &= \Big\{ A \in \mathbb{C}^{2\times 2} \ : \ A = -\overline{A}^{\perp}, \operatorname{tr}(A) = 0 \Big\}, \end{aligned}$$

where both have (real) dimension 3. Every matrix in $\mathfrak{so}(3)$ have the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

and therefore a basis of $\mathfrak{so}(3)$ is given by the three (linearly independent) matrices

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

satisfying the commutator relations

$$[\alpha_1, \alpha_2] = \alpha_3 \quad [\alpha_2, \alpha_3] = \alpha_1 \quad [\alpha_3, \alpha_1] = \alpha_2.$$

Every matrix in $\mathfrak{su}(2)$ has the form

$$\begin{pmatrix} ai & b+ci \\ -b+ci & -ai \end{pmatrix}$$

for $a, b \in \mathbb{R}$. Hence a basis is given by the three matrices

$$\beta_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \beta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \beta_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

satisfying the commutator relations

$$[\beta_1, \beta_2] = \beta_3 \quad [\beta_2, \beta_3] = \beta_1 \quad [\beta_3, \beta_1] = \beta_2.$$

Therefore $\Phi\colon \mathfrak{so}(3) \to \mathfrak{su}(2)$ defined by $\Phi(\alpha_1) = \beta_i$, $i = 1, 2, 3$ (and extended linearly) defines a Lie algebra isomorphism.

Alternative Solution. We show that S^3 and $\mathrm{SO}(3)$ have isomorphic Lie algebras; the strategy is to construct a smooth group homomorphism $S^3 \to \mathrm{SO}(3)$ with kernel $\{(1,0,0,0), (-1,0,0,0)\}$. In the following we will indentify $\mathbb{R}^3$ in $\mathbb{H}$ by considering the subspace of pure quaternions given by $\mathbb{R}^3 \cong \{(0,x,y,z) = xi + yj + zk | x, y, z \in \mathbb{R}\}$. Consider the group action of S^3 on $\mathbb{R}^3$ by linear maps given by

$$\psi_{(a,b,c,d)}(x,y,z) \coloneqq (a + bi + cj + dk)(xi + yj + zk)(a - bi - cj - dk).$$

One can show that this is indeed a homomorphism $\psi\colon S^3 \to \mathbb{R}^{3\times 3}$ and that $\psi_q \in \mathrm{GL}(3, \mathbb{R})$ for all $q \in S^3$. Moreover, for $q \in S^3$ and $w \in \mathbb{R}^3$, $\|\psi_q(w)\| = \|qwq^{-q}\| = \|q\|$ and therefore $\psi_q \in \mathrm{O}(s)$ for all $q \in S^3$. In fact, since S^3 is connected and the indentity is contained in the image of $\psi\colon S^3 \to \mathrm{O}(3)$, $q \mapsto \psi_q = \psi(q)$, it follows that $\psi_q \in \mathrm{SO}(3)$ for all $q \in S^3$. Thus ψ is a smooth group homomorphism and a computation shows that $\ker \psi = \{1_{\mathbb{H}}, -1_{\mathbb{H}}\}$, where $1_{\mathbb{H}} = (1,0,0,0)$ (this follows from the condition $\psi_q(e_i) = e_i$ for $i = 1,2,3$). Assume for the moment that ψ is surjective. Thus ψ induces an isomorphism from $S^3/\{1_{\mathbb{H}}, -1_{\mathbb{H}}\}$ to $\mathrm{SO}(3)$ and an argument similar to (a) show that every point in S^3 is regular. In particular $1_{\mathbb{H}}$ is regular and with the Inverse Function Theorem we find neighborhoods of $1_{\mathbb{H}}$ and $\mathbb{1}_{\mathrm{SO}(3)}$ on which ψ is a diffeomorphism preserving the group structure, and thus inducing an isomorphism of the corresponding Lie algebras.

We now show that ψ is surjective. Every matrix in $\mathrm{SO}(3)$ can be seen as a rotation in $\mathbb{R}^3$ around an axis trough the origin. We show that for every such rotation, there exists $(a,b,c,d) \in S^3$ such that $\psi_{(a,b,c,d)}$ realizes it.

Let $\mathbb{R}u \in \mathbb{R}^3 \subset \mathbb{H}$ be the rotation axis, where $u \in S^3$ and let $q = \cos(\alpha/2) + u\sin(\alpha/2)$ (we are identifying u with $(0, u_1, u_2, u_3)$ and scalars c with $(c,0,0,0)$). We show that

$$v' = qvq^{-q} = \big(\cos\frac{\alpha}{2} + u\sin\frac{\alpha}{2}\big)v\big(\cos\frac{\alpha}{2} - u\sin\frac{\alpha}{2}\big)$$

is the vector obtained by rotating v around $\mathbb{R}u$ by an angle α. Using the relation $uv = u \times v - u \cdot v$ (where $u \cdot v$ is the dot product of u and v as vectors in $\mathbb{R}^3$, seen in $\mathbb{R}^3 \subset \mathbb{H}$, and similarly for $u \times v$) we see that

$$\begin{aligned} v &= v\cos^2\frac{\alpha}{2} + (uv - vu)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - uvu\sin^2\frac{\alpha}{2} \\ &= v\cos^2\frac{\alpha}{2} + 2(u \times v)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}(v(u\cdot v) - 2u(u \cdot v))\sin^2\frac{\alpha}{2} \\ &= v(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}) + (u \times v)(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}) + u(u\cdot v)(2\sin^2\frac{\alpha}{2}) \\ &= v\cos\alpha + (u \times v)\sin\alpha + u(u \cdot v)(1 - \cos\alpha) \\ &= (v - u(u\cdot v))\cos\alpha + (u \times v)\sin\alpha + u(u \cdot v) \\ &= v_\perp \cos\alpha + (v \times v_\perp)\sin\alpha + v_\parallel, \end{aligned}$$

where $v_\perp and v_\parallel$ are the components of v perpendicular and parallel to u, respectively.

3. Exponential Map

Let $0 < \alpha \neq 1$. Show that there doesn't exist $A \in \mathbb{R}^{2\times 2}$ with

$$e^A = \begin{pmatrix} -\alpha & 0 \\ 0 & -\frac{1}{\alpha} \end{pmatrix}$$

and conclude that the exponential map of a connected Lie group is not necessarily surjective (for example, consider the Lie group $\mathrm{GL}^+(n, \mathbb{R})$).

Solution. Suppose there exists A with

$$a^A = \begin{pmatrix} -\alpha & 0 \\ 0 & -\frac{1}{\alpha} \end{pmatrix} =: B.$$

From the lecture, we know that $1 = \det B = \det(e^A) = e^{\mathrm{trace}(A)}$ and therefore $\mathrm{trace}(A) = 0$, so A has the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

In order to compute e^A we have to compute powers of A:

$$A^2 = \begin{pmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{pmatrix} = (a^2 + bc) \cdot \mathbb{1}.$$

We consider three cases.

If $a^2 + bc = 0$, then $e^A = \mathbb{1} + A$, which cannot be B since at least one of the terms $1 + a$ or $1 - a$ on the diagonal is positive.

If $a^2 + bc > 0$ let $\omega > 0$ with $\omega^2 = a^2 + bc$, so that $A^2 = \omega^2 \mathbb{1}$, $A^3 = \omega^2 A$, $A^4 = \omega^4 \mathbb{1}$, $A^5 = \omega^4 A$, and so on. Thus

$$\begin{aligned} e^A &= \sum_{j=0}^{\infty} \frac{\omega^{2j}}{(2j)!} \mathbb{1} + \frac{\omega^{2j}}{(2j+1)!} A \\ &= \sum_{j=0}^{\infty} \frac{\omega^{2j}}{(2j)!} \mathbb{1} + \frac{1}{\omega} \sum_{j=0}^{\infty} \frac{\omega^{2j+1}}{(2j+1)!} A \\ &= \cosh(\omega) \mathbb{1} + \frac{\sinh(\omega)}{\omega} A. \end{aligned}$$

Since both $\cosh(\omega)$ and $\sinh(\omega)$ are positive, by the same argument as before we conclude that e^A can't be B.

If $a^2 + bc < 0$ let $\omega > 0$ with $\omega^2 = -a^2 + bc$ so that $A^2 = -\omega^2 \mathbb{1}$, $A^3 = -\omega^2 A$, $A^4 = \omega^4 \mathbb{1}$, $A^5 = \omega^4 A$ and so on. hence

$$e^A = \sum_{j=0}^{\infty} (-1)^j \frac{\omega^{2j}}{(2j)!} \mathbb{1} + \frac{1}{\omega} \sum_{j=0}^{\infty} (-1)^j \frac{\omega^{2j+1}}{(2j+1)!} A = \cos(\omega) \mathbb{1} + \frac{\sin(\omega)}{\omega} A.$$

This implies that either $b = c = 0$ or $\sin(\omega) = 0$. $b = c = 0$ is not possible for otherwise $a^2 < 0$. Hence $\sin(\omega) = 0$ so $e^A = \cos(\omega)\mathbb{1}$ and $\alpha = \frac{1}{\alpha}$, which is not possible beacuse $\alpha \neq 1$.