Solutions 3

1. Bi-invariant Metrics

A Riemannian metric $\langle\cdot,\cdot\rangle$ on a Lie group G is called *bi-invariant* if for all $g \in G$ the left-translation $l_g \colon G \to G$, $l_g(x) := gx$, and the right-translation $r_g \colon G \to G$, $r_g(x) := xg$, are isometries.

a) Show that for $G = \mathrm{SO}(n, \mathbb{R})$, $TG_g = \{(g, gA) : \ A \in \mathbb{R}^{n\times n}, A = -A^{\mathrm{T}}\}$,

$$\langle (g, gA), (g, gB)\rangle := \frac{1}{2}\mathrm{tr}(AB^{\mathrm{T}})$$

defines a bi-invariant metric on G.

b) Show that every compact Lie group admits a bi-invariant metric.

Hint: Define first a left-invariant metric on G, then use an appropriate integration over G.

c) Let G be a Lie group with a bi-invariant metric and let D be the corresponding Levi-Civita connection. Prove that for left-invariant vector fields $X, Y \in \Gamma(TG)$ we have

$$D_X Y = \frac{1}{2}\,[X, Y]\,.$$

Solution. a) It was shown in class that $l_{h*}(g, gA) = (hg, hgA)$ and similarly one can show that $r_{h*}(g, gA) = (gh, gAh)$. $\langle\cdot,\cdot\rangle$ is left-invariant:

$$\begin{aligned}
(l_h^*\langle\cdot,\cdot\rangle)_g((g, gA), (g, gB)) &= \langle l_{h*}(g, gA), l_{h*}(g, gB)\rangle \\
&= \langle (hg, hgA), (hg, hgB)\rangle \\
&= \frac{1}{2}\mathrm{tr}(hgA(hgB)^{\mathrm{T}}) \\
&= \frac{1}{2}\mathrm{tr}(hgAB^{\mathrm{T}}(hg)^{\mathrm{T}}) \\
&= \frac{1}{2}\mathrm{tr}(AB^{\mathrm{T}}) \\
&= \langle (g, gA), (g, gB)\rangle.
\end{aligned}$$

$\langle\cdot,\cdot\rangle$ is right-invariant:

$$\begin{aligned}\langle r_{h*}(g,gA), r_{h*}(g,gB)\rangle &= \langle (gh,gAh),(gh,gBh)\rangle \\ &= \frac{1}{2}\mathrm{tr}(gAhh^{\mathrm{T}}B^{\mathrm{T}}g^{\mathrm{T}}) \\ &= \frac{1}{2}\mathrm{tr}(AB) \\ &= \langle (g,gA),(g,gB)\rangle.\end{aligned}$$

We have to show that $\langle\cdot,\cdot\rangle$ defines an inner product. Multilinearity and symmetry are immediate, so it remains to show positive definitness, that is, $\mathrm{tr}(AA^{\mathrm{T}}) > 0$ if $A \neq 0$. We have

$$\mathrm{tr}AA^{\mathrm{T}} = \sum_{i=1}^{n}(AA^{\mathrm{T}})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(A^{\mathrm{T}})_{ji} = \sum_{i,j=1}^{n} A_{ij}^2 \geq 0,$$

whith equality if and only if $A \equiv 0$.

b)Let $\langle\cdot,\cdot\rangle_e'$ be a scalar product on TG_e. We define a metric $\langle\cdot,\cdot\rangle'$ on G as follows. For $X, Y \in TG_g$ we set

$$\langle X,Y\rangle_g' := \langle d(L_{g^{-1}})_g X, d(L_{g^{-1}})_g Y\rangle_e'.$$

Note that $\langle\cdot,\cdot\rangle'$ is left-invariant.

Now, assume that G has dimension n. Choose a non-zero alternating multilinear form $\omega_0 \in \Lambda^n(TG_e^*)$ and extend it to the whole G via left-multiplication: for $g \in G$ and $v_1,\dots,v_n \in TG_g$ set

$$\omega_g(v_1,\dots,v_n) := \omega_0\big(d(L_{g^{-1}})_g(v_1),\dots,d(L_{g^{-1}})_g(v_n)\big).$$

One can check that $\omega \in \Omega^n(G)$ and that it's left-invariant, that is, $L_g^*\omega = \omega$ for all $g \in G$. We endow G with an orientation by declaring a basis $e_1,\dots,e_n$ of TG_g positive if $\omega_g(e_1,\dots,e_n) > 0$. With this orientation left-multiplications are orientation preserving isometries.

We define an operator $I\colon C^\infty(G) \to \mathbb{R}$ by setting

$$I(f) := \int_G f\omega := \int_G f\,\mathrm{d}V := \int_G f(h)\,\mathrm{d}V(h).$$

Observe that for $g \in G$, $I(f \circ L_g) = I(f)$.

We define $\langle\cdot,\cdot\rangle$ on G by integrating $\langle\cdot,\cdot\rangle'$ with I as follows. For $p \in G$ and $X, Y \in TG_p$ we set

$$\langle X,Y\rangle_p := \int_G \langle (dR_h)_p(X),(dR_h)_p(Y)\rangle_{ph}'\omega = I(f)$$

for $f(h) := \langle d(R_h)_p(X), d(R_h)_p(Y)\rangle'_{ph}$. Left-invariance follows by left-invariance of $\langle\cdot,\cdot\rangle'$ and the fact that $R_h \circ L_g = L_g \circ R_h$. We now show right-invariance: let $p, h \in G$ and $X, Y \in TG_p$, then

$$\begin{aligned}(R_g^*\langle\cdot,\cdot\rangle)_p(X,Y) &= \langle (dR_g)_p(X), (dR_g)_p(Y)\rangle_{pg} \\ &= \int_G \langle (dR_h)_{pg}(dR_g)_p(X), \dots\rangle'_{pgh}\omega \\ &= \int_G \langle d(R_h \circ R_g)_p(X), \dots\rangle'_{pgh}\omega \\ &= \int_G \underbrace{\langle d(R_{gh})_p(X), \dots\rangle'_{pgh}}_{=f(gh)=f\circ L_g(h)}\omega \\ &= I(f\circ L_g) = I(f) \\ &= \int_G \langle d(R_h)_p(X), d(R_h)_p(Y)\rangle'_{ph}\omega \\ &= \langle X, Y\rangle_p.\end{aligned}$$

Symmetry and positive-definitness are immediate.

c) Before solving the exercise we need some additional machinery.

For every $g \in G$ consider the conjugation map $C_g := R_{g^{-1}} \circ L_g \colon G \to G$, $C_g(x) = gxg^{-1}$. It is a Lie group isomorphism hence it's differential at the identity is a Lie algebra isomorphism: for $g \in G$ we define the adjoint map $\mathrm{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$, $\mathrm{Ad}(g) := d(C_g)_e$. Note that this defines a map $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ and hence $(d\mathrm{Ad})_e\colon \mathfrak{g} \to T(\mathrm{GL}(\mathfrak{g}))_e = \mathfrak{gl}(\mathfrak{g})$, which can be also written

$$(d\mathrm{Ad})_e(X_e)(Y_e) = \frac{d}{dt}\Big|_{t=0} \exp(tX_e)(Y_e),$$

for $X_e, Y_e \in \mathfrak{g}$.

Claim (A). Given left-invariant vector fields $X, Y \in \Gamma(TG)$ we have

$$d\mathrm{Ad}_e(X_e)(Y_e) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}\big(\exp(tX_e)\big)(Y_e) = [X_e, Y_e] = [X,Y]_e\,.$$

Claim (B). If $\langle\cdot,\cdot\rangle$ is bi-invariant, then for all $X_e, Y_e, Z_e \in \mathfrak{g}$

$$\langle [X_e, Y_e], Z_e\rangle + \langle Y_e, [X_e, Z_e]\rangle = 0.$$

Let's assume Claim (A) and (B) for the moment.

Let X, Y, Z be left-invariant vector fields.

Note that by left-invariance of the vector fields and of the metric we have

$$\langle X_p, Y_p\rangle_p = \langle (dL_p)_e X_e, (dL_p)_e Y_e\rangle_p = \langle X_e, Y_e\rangle_e.$$

In particular the function $p \mapsto \langle X_p, Y_p \rangle_p$ is constant and the same holds for all pair of left-invariance vector fields. This means that the Koszul formula in this case gives

$$2\langle D_X Y, Z\rangle = -\langle X, [Y,Z]\rangle - \langle Y, [X,Z]\rangle + \langle Z, [X,Y]\rangle.$$

It follows by Claim (B) that $-\langle X_e, [Y_e, Z_e]\rangle_e - \langle Y_e, [X_e, Z_e]\rangle_e = 0$, so $-\langle X, [Y,Z]\rangle - \langle Y, [X,Z]\rangle = 0$ and therefore $2\langle D_X Y, Z\rangle = \langle Z, [X,Y]\rangle$. As this holds for all left-invariant Z this proves the desired result.

Proof of Claim (A). As a preliminary step note that for $X \in \Gamma(TG)$ left-invariant, $f \in C^\infty(G)$ and $g \in G$

$$X(f)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp tX_e).$$

Now, using this formula twice we compute

$$\begin{aligned}&[X,Y]\,(f)(g)\\ &\quad = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} f(g\exp(tX_e)\exp(sY_e)) - \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} f(g\exp(sY_e)\exp(tX_e)).\end{aligned}$$

By reversing the order of differentiation in the first term and composing with $t \mapsto -t$ in the second term, this expression becomes

$$\begin{aligned}&\frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} f(g\exp(tX_e)\exp(sY_e)) - f(g\exp(sY_e)\exp(-tX_e))\\ &\quad = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} f\big(g\exp(tX_e)\exp(sY_e)\exp(-tX_e)\big).\end{aligned}$$

Where we have used that

$$\frac{d}{dt}\Big|_{t=0} F(t,t) = \frac{d}{dt}\Big|_{t=0} F(t,0) + \frac{d}{dt}\Big|_{t=0} F(0,t)$$

applied to the map $F(x,y) := f(g\exp(xX_e)\exp(sY_e)\exp(-yX_e))$.

Thus using the above formula again we see that

$$\begin{aligned}&\frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} f\big(g\exp(tX_e)\exp(sY_e)\exp(-tX_e)\big)\\ &\quad = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} f\Big(g\exp\Big(s\mathrm{Ad}(\exp(tX_e))Y_e\Big)\Big)\\ &\quad = \Big(\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(\exp(tX_e))Y_e\Big)(f)(g)\end{aligned}$$

where we have used that the map $Z \mapsto Z(f)(g)$ is linear. $\square$

Proof of Claim (B). Note that since $\langle\cdot,\cdot\rangle$ is bi-invariant we have for $g \in G$ and $X_e, Y_e \in \mathfrak{g}$ $\langle \mathrm{Ad}(g)X_e, \mathrm{Ad}(g)Y_e\rangle_e = \langle X_e, Y_e\rangle$. Applying this to $g = \exp(tX_e)$ we see that

$$t \mapsto \langle \mathrm{Ad}(\exp(tX_e))Y_e, \mathrm{Ad}(\exp(tX_e))Z_e\rangle_e$$

is constant and therefore by Claim (A)

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} \langle \mathrm{Ad}(\exp(tX_e))Y_e, \mathrm{Ad}(\exp(tX_e))Z_e\rangle_e \\
&= \langle \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX_e))Y_e, Z_e\rangle_e + \langle Y_e, \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX_e))Z_e\rangle_e \\
&= \langle [X_e, Y_e], Z_e\rangle + \langle Y_e, [X_e, Z_e]\rangle.
\end{aligned}$$

□

2. The Levi-Civita connection on a submanifold

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with Levi-Civita connection $\bar{D}$, and let M be a submanifold of $\bar{M}$, equipped with the induced metric $g := i^*\bar{g}$, where $i\colon M \to \bar{M}$ is the inclusion map.

Show that the Levi-Civita connection D of (M, g) satisfies $D_X Y = (\bar{D}_X Y)^{\mathrm{T}}$ for all $X, Y \in \Gamma(TM)$, where the superscript T denotes the component tangential to M and $\bar{D}_X Y$ is defined(!) as $\bar{D}_X Y := \bar{D}_{\bar{X}} \bar{Y}$ for any extensions $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$ of X, Y.

Solution. As we have seen in the lecture (Remark 1.7), that $(\bar{D}_{\bar{X}}\bar{Y})_p$ only depends on $\bar{X}_p$ and $\bar{Y} \circ c$, where $c\colon (-\epsilon, \epsilon) \to \bar{M}$ is a curve with $\dot{c}(0) = \bar{X}$. Hence $\bar{D}_X Y$ is independent of the choice of the extensions $\bar{X}$ and $\bar{Y}$.

Clearly, $(\bar{D}_X Y)^{\mathrm{T}}$ defines a linear connection. It remains to prove that this connection is compatible with g and torsion-free. For $X, Y, Z \in TM$, we have

$$\begin{aligned}
Zg(X,Y) &= \bar{Z}\bar{g}(\bar{X}, \bar{Y}) = \bar{g}(\bar{D}_{\bar{Z}}\bar{X}, \bar{Y}) + \bar{g}(\bar{X}, \bar{D}_{\bar{Z}}\bar{Y}) \\
&= \bar{g}((\bar{D}_Z X)^{\mathrm{T}}, \bar{Y}) + \bar{g}(\bar{X}, (\bar{D}_Z Y)^{\mathrm{T}}) = g((\bar{D}_Z X)^{\mathrm{T}}, Y) + g(X, (\bar{D}_Z Y)^{\mathrm{T}})
\end{aligned}$$

and

$$(\bar{D}_X Y) - (\bar{D}_Y X) = (\bar{D}_{\bar{X}}\bar{Y})^{\mathrm{T}} - (\bar{D}_{\bar{Y}}\bar{X})^{\mathrm{T}} = [\bar{X}, \bar{Y}]^{\mathrm{T}} = [X, Y].$$

3. Gradient and Hessian form

Let (M, g) be a Riemannian manifold, D the Levi-Civita connection and $f\colon M \to \mathbb{R}$ a smooth function on M.

a) The *gradient* $\mathrm{grad} f \in \Gamma(TM)$ is defined by

$$df(X) = g(\mathrm{grad} f, X), \quad \forall X \in \Gamma(TM).$$

Compute $\mathrm{grad} f$ in local coordinates.

b) The *Hessian form* $\mathrm{Hess}(f) \in \Gamma(T_{0,2}M)$ is defined by

$$\mathrm{Hess}(f)(X, Y) = g(D_X \mathrm{grad} f, Y), \quad \forall X, Y \in \Gamma(TM).$$

Prove that $\mathrm{Hess}(f)$ is symmetric and compute $\mathrm{Hess}(f)$ in local coordinates.

Solution. a) For a chart (ϕ, U), let $A_i := \frac{\partial}{\partial \phi^i}$ and $\mathrm{grad} f = \sum_i Y^i A_i$. Then we have

$$\begin{aligned} f_j := \tfrac{\partial}{\partial \phi^j}(f) &= df\,(A_j) = g\,(\mathrm{grad} f, A_j) \\ &= g\left(\sum_i Y^i A_i, A_j\right) = \sum_i Y^i g\,(A_i, A_j) = \sum_i Y^i g_{ij} \end{aligned}$$

Hence we get $Y^i = \sum_j f_j g^{ji}$ and thus $\mathrm{grad} f = \sum_{i,j} g^{ji} f_j A_i$.

b) First, we use that D is compatible with g. We get

$$\begin{aligned} \mathrm{Hess}(f)(X, Y) &= g(D_X \mathrm{grad} f, Y) = Xg(\mathrm{grad} f, Y) - g(\mathrm{grad} f, D_X Y) \\ &= X(Y(f)) - (D_X Y)(f) \end{aligned}$$

Since the Levi-Civita connection D is torsion free, it follows

$$\begin{aligned} \mathrm{Hess}(f)(X, Y) &= X(Y(f)) - (D_X Y)(f) + T(X, Y)(f) \\ &= Y(X(f)) - (D_Y X)(f) = \mathrm{Hess}(f)(Y, X), \end{aligned}$$

i.e. $\mathrm{Hess}(f)$ is symmetric. Furthermore, we get in local coordinates

$$\mathrm{Hess}(f)_{ij} = \mathrm{Hess}(f)(A_i, A_j) = A_i(A_j(f)) - (D_{A_i} A_j)(f) = f_{ij} - \sum_k \Gamma^k_{ij} f_k$$