Solutions 4

1. Existence of closed geodesics

Let (M, g) be a compact Riemannian manifold and $c_0\colon S^1 \to M$ a continuous closed curve. The purpose of this exercise is to show that in the family of all continuous and piece-wise C^1 curves $c\colon S^1 \to M$ which are homotopic to c_0, there is a shortest one and it is a geodesic.

a) Show that c_0 is homotopic to a piece-wise C^1-curve c_1 with finite length.

b) Let $L := \inf_c L(c)$ be the infimum over all piece-wise C^1 curves $c\colon S^1 \to M$ homotopic to c_0 and consider a minimizing sequence $(c_n\colon S^1 \to M)_n$ with $\lim_n L(c_n) = L$. Use compactness of M to construct a piece-wise C^1-curve $c\colon S^1 \to M$ with length L.

   Hint. Cover M with simply connected balls with the property that every two points in a ball are joined by a unique distance minimizing geodesic.

c) Conclude by showing that c is homotopic to c_0 and a geodesic.

Solution. a) Let us first prove that c_0 is homotopic to a piece-wise C^1-curve c_1. To this aim, we split c_0 into finitely many paths $\gamma_i\colon [0,1] \to M$ such that $\gamma_i(1) = \gamma_{i+1}(0)$, $\gamma_n(1) = \gamma_1(0)$ and γ_i is contained in a charts $\{(\phi_i, U_i)\}_{i=1}^n$ with U_i simply-connected. Then γ_i is homotopic (relative to the endpoints) to a C^1-curve $\widetilde{\gamma}_i$ and by connecting the $\widetilde{\gamma}_i$'s we get a piece-wise C^1-curve c_1 which is homotopic to c_0. Then c_1 has finite length $L(c_1)$.

b) Let $L := \inf_c L(c) < \infty$ be the infimum over all curves $c\colon S^1 \to M$ which are piece-wise C^1 and homotopic to c_0 and consider a minimizing sequence, i.e. a sequence $(c_n\colon S^1 \to M)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} L(c_n) = L$.

We may assume that the curves $c_n\colon [0,1] \to M$ are parametrized proportional to arclength, i.e. $L(c_n|_{[a,b]}) = |b-a| \cdot L(c_n)$.

As M is compact, there is some $r > 0$ and points $q_q, \dots, q_n \in M$ such that the balls $B(q_1, r), \dots, B(q_n, r)$ cover M, for all $q, q' \in B(q_i, 3r)$ there is a unique distance minimizing geodesic joining q to q' of length $< 6r$ and the balls $B(q_i, 6r)$ are simply connected.

Fix some $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{r}{L}$ and define $t_k := \frac{k}{N}$ for $k = 0, \dots, N$. Consider now the sequences $(c_n(t_k))_{n\in\mathbb{N}}$. By compactness of M, we may assume (by possibly passing to subsequences) that $c_n(t_k) \to p_k$ for each $k = 0, \dots, N$. Therefore

$$d(p_k, p_{k+1}) \le \limsup_{n\to\infty} d(c_n(t_k), c_n(t_{k+1})) \le \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) < r.$$

Take $q \in \{q_1, \dots, q_n\}$ such that $p_k \in B(q, r)$, then $p_{k+1} \subset B(q, 3r)$ and therefore we can define a continuous, piece-wise C^1-curve $c\colon [0,1] \to M$ by concatenating the unique distance minimizing geodesics between p_k and p_{k+1}.

For the length of c we have

$$L(c) = \sum_{k=0}^{N-1} L\left(c|_{[t_k, t_{k+1}]}\right) = \sum_{k=0}^{N-1} d(p_k, p_{k+1}) \leq N \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) = L.$$

c) It remains to prove that c is homotopic to c_0. Observe that for n large enough, we have $c([t_k, t_{k+1}]), c_n([t_k, t_{k+1}]) \subset B(q, 3r)$.

Since $B(q, 6r)$ is simply-connected there is a homotopy from $c_n|_{[\frac{k}{N}, \frac{k+1}{N}]}$ to $c|_{[\frac{k}{N}, \frac{k+1}{N}]}$ with the endpoints following the unique geodesics from $c_n(t_k)$ to p_k and from $c_n(t_{k+1})$ to p_{k+1}, respectively. Combining this homotopies, we get a homotopy from c_n to c.

Observe that c is locally length minimizing and hence is a geodesic.

2. Metric and Riemannian isometries

Let (M, g) and $(\bar{M}, \bar{g})$ be two connected Riemannian manifolds with induced distance functions d and $\bar{d}$, respectively. Further, let $f\colon (M, d) \to (\bar{M}, \bar{d})$ be an isometry of metric spaces, i.e. f is surjective and for all $p, p' \in M$ we have $\bar{d}(f(p), f(p')) = d(p, p')$.

a) Prove that for every geodesic γ in M, $\bar{\gamma} := f \circ \gamma$ is a geodesic in N.

b) Let $p \in M$. Define $F\colon TM_p \to T\bar{M}_{f(p)}$ with

$$F(X) := \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma_X(t),$$

where γ_X is the geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}(0) = X$. Show that F is surjective and satisfies $F(cX) = cF(X)$ for all $X \in TM_p$ and $c \in \mathbb{R}$.

c) Conclude that F is an isometry by proving $\|F(X)\| = \|X\|$.

d) Prove that F is linear and conclude that f is smooth in a neighborhood of p.

e) Prove that f is a diffeomorphism for which $f^*\bar{g} = g$ holds.

Solution. a) As the property of being a geodesic is local, we may assume that both $\gamma\colon [0, L] \to M$ and $f \circ \gamma\colon [0, L] \to \bar{M}$ are contained in an open set

$U \subset M$ and $\bar{U} \subset \bar{M}$, respectively, such that points in U and $\bar{U}$ are connected by a unique distance minimizing geodesic. Then there is a unique geodesic β from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$. We claim that $\bar{\gamma}$ and β coincide.

In the following all geodesics are parametrized by arclength. For $t \in [0, L]$ there are geodesics β_1 from $\bar{\gamma}(0)$ to $\bar{\gamma}(t)$ and β_2 from $\bar{\gamma}(t)$ to $\bar{\gamma}(L)$. Concatenating β_1 and β_2, we get some piece-wise C^1-curve from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$ with length

$$\begin{aligned} L(\beta_1\beta_2) &= L(\beta_1) + L(\beta_2) \\ &= \bar{d}(\bar{\gamma}(0), \bar{\gamma}(t)) + \bar{d}(\bar{\gamma}(t), \bar{\gamma}(L)) \\ &= d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(L)) \\ &= d(\gamma(0), \gamma(L)) = \bar{d}(\bar{\gamma}(0), \bar{\gamma}(L)) = L(\beta). \end{aligned}$$

Hence, by uniqueness of the geodesic from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$, $\beta_1\beta_2$ and β coincide, i.e. $\bar{\gamma}(t) = \beta(t)$.

b) Observe that f is bijective and its inverse f^{-1} is also is an isometry of metric spaces.

First, we prove that F is surjective. Let $Y \in T\bar{M}_{f(p)}$ and $\bar{\gamma}$ the geodesic through $f(p)$ with $\dot{\bar{\gamma}}(0) = Y$. Then $Y = F(X)$ for $X := \frac{d}{dt}\big|_{t=0} f^{-1} \circ \bar{\gamma}(t)$.

From $\gamma_{cX}(t) = \gamma_X(ct)$ it follows that

$$F(cX) = \frac{d}{dt}\bigg|_{t=0} f \circ \gamma_X(ct) = cF(X).$$

c) For $\epsilon > 0$ small enough, we have that $\gamma_X(\epsilon)$ and $f \circ \gamma_X(\epsilon)$ are contained in a normal neighborhood of p and $f(p)$, respectively. Hence we get

$$\epsilon\|X\| = d(p, \gamma_X(\epsilon)) = \bar{d}(f(p), f \circ \gamma_X(\epsilon)) = \epsilon\|F(X)\|.$$

We now claim that for $X, Y \in TM_p$ with $\|X\| = \|Y\| = 1$ and α such that $\cos\alpha = g_p(X, Y)$ we have

$$\sin\frac{1}{2}\alpha = \lim_{s\to 0} \frac{1}{2s} d(\gamma_X(s), \gamma_Y(s)),$$

and a similar formula for $\bar{X}, \bar{Y} \in T\bar{M}_f(p)$ with $\|\bar{X}\| = \|\bar{Y}\| = 1$.

Assuming the claim for the moment, we now prove that

$$g_p(X, Y) = \bar{g}_{f(p)}(F(X), F(Y)).$$

for all $X, Y \in TM_p$.

Note first that since $F(cX) = cF(X)$, we can assume that $\|X\| = \|Y\| = 1$, then also $\|F(X)\| = \|F(Y)\| = 1$. So by the claim and the fact that

f is a distance preserving map we have for $\cos\alpha = g_p(X,Y)$ and $\cos\alpha' = \bar{g}_{f(p)}(F(X),F(Y))$

$$\sin\frac{1}{2}\alpha = \sin\frac{1}{2}\alpha'.$$

Therefore

$$g_p(X,Y) = \cos\alpha = 1 - 2\sin^2\frac{1}{2}\alpha = 1 - 2\sin^2\frac{1}{2}\alpha' = \bar{g}_{f(p)}(F(X),F(Y)).$$

d)For all $X,Y,Z \in TM_p$ and $c \in \mathbb{R}$, we have

$$\begin{aligned}\bar{g}_{f(p)}(F(X+cY),F(Z)) &= g_p(X+cY,Z)\\ &= g_p(X,Z) + cg_p(Y,Z)\\ &= \bar{g}_{f(p)}(F(X),F(Z)) + c\bar{g}_{f(p)}(F(Y),F(Z))\\ &= \bar{g}_{f(p)}(F(X)+cF(Y),F(Z))\end{aligned}$$

Hence F is linear and therefore smooth.

If V_p is a neighborhood of $0 \in TM_p$ such that $\exp_p|_{V_p}\colon V_p \to U_p$ is a diffeomorphism, then we have

$$f|_{U_p} = \exp_{f(p)} \circ F \circ (\exp_p|_{V_p})^{-1}.$$

Hence f is smooth as well.

e) The argument above works for all $p \in M$ and also for f^{-1}. Hence f is a diffeomorphism. Furthermore, we have

$$df_p = d(\exp_{f(p)} \circ F \circ \exp_p^{-1}) = F$$

and thus

$$f^*\bar{g}_p(X_p,Y_p) = \bar{g}_{f(p)}(df_p(X_p),df_p(Y_p)) = \bar{g}_{f(p)}(F(X_p),F(Y_p)) = g_p(X_p,Y_p),$$

for all $X,Y \in TM$.

Proof of the claim (sketch). Let $X,Y \in TM_p$ with $\|X\| = \|Y\| = 1$ and let $\alpha = \sphericalangle_0(X,Y)$, that is, $\cos\alpha = g_p(X,Y)$. Consider normal coordinates $(\varphi, B(p,r))$ around p, so that we have $\varphi\colon B(p,r) \to B_r \subset \mathbb{R}^n$ and define $c_X \coloneqq \varphi\circ\gamma_X$ and $c_Y \coloneqq \varphi\circ\gamma_Y$, two curves in B_r.

On B_r we can consider two different metrics. The Euclidean metric g_E and the pull-back metric $h \coloneqq (\varphi^{-1})^*g$.

Note that $h_0(c_X'(0),c_Y'(0)) = g_p(X,Y)$ and by Lemma 1.19 $h_0 = (g_E)_0$, so $(g_E)_0(c_X'(0),c_Y'(0)) = g_p(X,Y)$. We are now in a completely Euclidean setting.

Suppose by contradiction that $\limsup_{s\to 0} \frac{1}{2s} d(\gamma_X(s), \gamma_Y(s)) > \sin \frac{1}{2}\alpha$ and take $c > 1$ such that

$$\limsup_{s\to 0} \tfrac{1}{2s} d(\gamma_X(s), \gamma_Y(s)) > c \sin \tfrac{1}{2}\alpha.$$

Now, take r small enough such that $c^{-1} \cdot g_E < h < c \cdot g_E$ on $B_r \subset \mathbb{R}^n$, and therefore

$$c^{-1} \cdot d_E < d_h < c \cdot d_E,$$

where d_h denotes the distance function induced by the metric h. This implies that for s small enough

$$\frac{1}{2s} d_E(c_X(s), c_Y(s)) > c \sin \frac{1}{2}\alpha,$$

which is not true. The other inequality follows similarly. $\square$

3. Homogeneous Riemannian manifolds

Let (M, g) be a *homogeneous Riemannian manifold*, i.e. the isometry group of M acts transitively on M. Prove that M is geodesically complete.

Solution. Let $p \in M$. Pick $r > 0$ such that $\exp_p$ is defined on $B(0,r) \subset TM_p$. In particular $[-1,1] \to M$, $t \mapsto \exp_p(tv)$ is defined for all $v \in B(0,r)$. For any $q = \exp_p(t_0)$ there exists an isometry Φ and $w \in B(0,r)$ such that $\Phi(p) = q$ and $d\Phi_p(w) = v$. So $\Phi \circ \exp_p(tw)$ extends $\exp_p(tv)$ on $[-1,1] \cup (t_0 - 1, t_0 + 1)$.

This shows that $\exp_p(tv)$ is defined on $(-\infty, \infty)$ and therefore M is geodesically complete.

Remark. By the Theorem of Hopf-Rinow this implies that M is complete.