Solutions 5

1. Constant sectional curvature

Let (M, g) be a Riemannian manifold with constant sectional curvature $\sec(E) = \kappa \in \mathbb{R}$ for all $E \in G_2(TM)$. Show that

$$R(X,Y)W = \kappa\,(g(Y,W)X - g(X,W)Y)\,.$$

Solution. As the sectional curvature is constant, we have

$$R(X,Y,X,Y) = \kappa\,(g(X,X)g(Y,Y) - g(X,Y)g(X,Y))$$

for all $X, Y \in \Gamma(TM)$. Consider now the $(0,4)$-tensor T given by

$$T(V,W,X,Y) \coloneqq \kappa\,(g(V,X)g(Y,W) - g(V,Y)g(X,W))\,.$$

Then the $(0,4)$-tensor $S \coloneqq R - T$ has the following symmetry properties:

1. $S(V,W,X,Y) = -S(V,W,Y,X)$,
2. $S(V,W,X,Y) + S(V,Y,W,X) + S(V,X,Y,W) = 0$,
3. $S(V,W,X,Y) = S(X,Y,V,W)$,
4. $S(X,Y,X,Y) = 0$.

The first three properties hold for R and T, while the last one was already observed above. Our goal is now to show that $S \equiv 0$.

For all $A, B, C, D \in \Gamma(TM)$, we have by 3. and 4.

$$\begin{aligned}0 &= S(A, B+D, A, B+D)\\ &= S(A,B,A,B) + S(A,B,A,D) + S(A,D,A,B) + S(A,D,A,D)\\ &= 2S(A,B,A,D)\end{aligned}$$

and

$$\begin{aligned}0 &= S(A+C, B, A+C, D)\\ &= S(A,B,A,D) + S(A,B,C,D) + S(C,B,A,D) + S(C,B,C,D)\\ &= S(A,B,C,D) + S(A,D,C,B).\end{aligned}$$

Finally, we get

$$\begin{aligned}3S(V,W,X,Y) &= S(V,W,X,Y) - S(V,Y,X,W) - S(V,W,Y,X)\\ &= S(V,W,X,Y) + S(V,Y,W,X) + S(V,X,Y,W) = 0,\end{aligned}$$

for all $V, W, X, Y \in \Gamma(TM)$.

2. Ricci curvature

Let (M, g) be a 3-dimensional Riemannian manifold. Show the following:

a) The Ricci curvature ric uniquely determines the Riemannian curvature tensor R.

b) If M is an Einstein manifold, that is, a Riemannian manifold (M, g) with $\mathrm{ric} = kg$ for some $k \in \mathbb{R}$, then the sectional curvature sec is constant.

Solution. a) In the following, let e_1, e_2, e_3 be an orthonormal basis of TM_p. First, note that $R_{iijk} = R_{jkii} = 0$ by the symmetry properties of R.

We denote the components of ric by R_{ij}. Then, for $\{i, j, k\} = \{1, 2, 3\}$, we have

$$\begin{aligned} R_{ii} &= R_{iiii} + R_{jiji} + R_{kiki} = R_{ijij} + R_{ikik}, \\ R_{ij} &= R_{iiij} + R_{jijj} + R_{kikj} = R_{ikjk} \end{aligned}$$

and therefore, we get

$$\begin{aligned} 2R_{ijij} &= R_{ii} + R_{jj} - R_{kk}, \\ R_{ikjk} &= R_{ij}. \end{aligned}$$

Observe now, that we can compute all other components of R by symmetry properties. Hence R is uniquely determined by ric.

b) Let e_1, e_2 be a orthonormal basis of $E \subset TM_p$ and choose e_3 such that e_1, e_2, e_3 is an orthonormal basis of TM_p. Then we have

$$2 \sec_p(E) = 2R_{1212} = R_{11} + R_{22} - R_{33} = k + k - k = k$$

and hence $\sec_p(E) = \frac{k}{2}$.

3. Divergence and Laplacian

Let (M, g) be a Riemannian manifold with Levi-Civita connection D. The *divergence* $\mathrm{div}(Y)$ of a vector field $Y \in \Gamma(TM)$ is the contraction of the $(1,1)$-tensor field $DY \colon X \mapsto D_X Y$ and the *Laplacian* $\Delta \colon C^\infty(M) \to C^\infty(M)$ is defined by $\Delta f := \mathrm{div}(\mathrm{grad}\, f)$. Show that:

a) $\mathrm{div}(fY) = Y(f) + f \,\mathrm{div}\, Y$,

b) $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \mathrm{grad}\, f, \mathrm{grad}\, g\rangle$,

c) Compute Δf in local coordinates.

Solution. a) Let $p \in M$ and let $e_1, \dots, e_n$ be a orthonormal basis of TM_p. Then we have

$$\begin{aligned}
\mathrm{div}_p(fY) &= \sum_{i=1}^n \langle D_{e_i}(fY), e_i\rangle \\
&= \sum_{i=1}^n \langle e_i(f) Y_p + f(p) D_{e_i} Y, e_i\rangle \\
&= \sum_{i=1}^n e_i(f)\langle Y_p, e_i\rangle + \sum_{i=1}^n f(p)\langle D_{e_i}Y, e_i\rangle \\
&= Y_p(f) + f(p)\,\mathrm{div}_p(Y) \\
&= (Y(f) + f\,\mathrm{div}(Y))(p)
\end{aligned}$$

and hence $\mathrm{div}(fY) = Y(f) + f\,\mathrm{div}\, Y$.

b) First, recall the definition of $\mathrm{grad}\, f$ from Exercise Sheet 3, i.e. $X(f) = \langle \mathrm{grad}\, f, X\rangle$ and note that

$$\langle \mathrm{grad}(fg), X\rangle = X(fg) = X(f)g + fX(g) = \langle \mathrm{grad}(f)g + f\,\mathrm{grad}(g), X\rangle,$$

for all $X \in \Gamma(M)$ and thus $\mathrm{grad}(fg) = \mathrm{grad}(f)g + f\,\mathrm{grad}(g)$.

Therefore, we get

$$\begin{aligned}
\Delta(fg) &= \mathrm{div}(\mathrm{grad}(fg)) \\
&= \mathrm{div}(\mathrm{grad}(f)g + f\,\mathrm{grad}(g)) \\
&= \mathrm{div}(\mathrm{grad}\, f)g + \mathrm{grad}(f)(g) + f\,\mathrm{div}(\mathrm{grad}(g)) + \mathrm{grad}(g)(f) \\
&= \Delta(f)g + f\Delta(g) + 2\langle \mathrm{grad}\, f, \mathrm{grad}\, g\rangle.
\end{aligned}$$

c) In the following, we use Einstein notation. From Exercise Sheet 3, we already know that

$$\begin{aligned} \operatorname{grad} f &= g^{ik} f_i A_k, \\ D_{A_i} Y &= (A_i(Y^k) + Y^j \Gamma^k_{ij}) A_k. \end{aligned}$$

Therefore, we have

$$\operatorname{div}(Y) = A_k(Y^k) + Y^j \Gamma^k_{kj}.$$

Hence, we get

$$\begin{aligned} \Delta f &= \operatorname{div}(\operatorname{grad} f) \\ &= A_k(g^{ik} f_i) + g^{ij} f_i \Gamma^k_{kj}. \end{aligned}$$

With $G := \det(g_{ij})$, we can simplify this as follows. We have by the Jacobi formula and the chain rule

$$\Gamma^k_{kj} = \frac{1}{2} g^{kl} A_j(g_{kl}) = \frac{1}{2G} A_j(G) = \frac{1}{\sqrt{G}} A_j(\sqrt{G})$$

and therefore

$$\begin{aligned} \Delta f &= A_j(g^{ij} f_i) + g^{ij} f_i \frac{1}{\sqrt{G}} A_j(\sqrt{G}) \\ &= \frac{1}{\sqrt{G}} A_j \left(\sqrt{G} g^{ij} f_i \right). \end{aligned}$$