Solutions 6

1. Curvature of bi-invariant metrics

Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$ and Riemann curvature tensor R. Let $X, Y, V, W \in \Gamma(TG)$ be left-invariant vector fields. Show that

a) $R(X,Y)W = \frac{1}{4}[W, [X, Y]]$.

b) $\langle V, R(X,Y)W \rangle = R(V, W, X, Y) = \frac{1}{4}\langle [V, W], [X, Y] \rangle$,
in particular the sectional curvature is non-negative.

Solution. a) Let $X, Y, W \in \Gamma(TG)$ be left-invariant vector fields. From Exercise 3 of Sheet 3 we know that $D_X Y = \frac{1}{2}[X, Y]$ and Exercise 1b) of Sheet 2 implies that $D_X Y$ is left-invariant as well. Analogous considerations hold for the other tuples of vector fields involved. Hence we compute

$$\begin{aligned} R(X,Y)W &= D_X D_Y W - D_Y D_X W - D_{[X,Y]} W \\ &= \frac{1}{2} D_X [Y, W] - \frac{1}{2} D_Y [X, W] - \frac{1}{2} [[X, Y], W] \\ &= \frac{1}{4}[X, [Y, W]] - \frac{1}{4}[Y, [X, W]] + \frac{1}{2}[W, [X, Y]] \\ &= \frac{1}{4}[X, [Y, W]] + \frac{1}{4}[Y, [W, X]] + \frac{1}{2}[W, [X, Y]]\,. \end{aligned}$$

From the Jacobi identity we obtain $[X, [Y, W]] + [Y, [W, X]] = -[W, [X, Y]]$, which implies the result.

b) From a) and Exercise 3 of Sheet 3 we get $R(X,Y)W = \frac{1}{2} D_W [X, Y]$, so we compute

$$\begin{aligned} \langle V, R(X,Y)W \rangle &= \frac{1}{2}\langle V, D_W [X, Y] \rangle \\ &= \frac{1}{2} W \langle V, [X, Y] \rangle - \frac{1}{2}\langle D_W V, [X, Y] \rangle \\ &= -\frac{1}{4}\langle [W, V], [X, Y] \rangle \\ &= \frac{1}{4}\langle [V, W], [X, Y] \rangle, \end{aligned}$$

where we have used that $\langle V, [X, Y] \rangle$ is constant and D is compatible with $\langle \cdot, \cdot \rangle$.

Now, let $p \in G$, $E \subset TG_p$ a 2-dimensional subspace and u, v an orthonormal basis for E. Denote by $U, V \in \Gamma(TG)$ the left-invariant vector fields with $U_p = u$ and $V_p = v$, respectively. Then

$$\sec_p(E) = R_p(u,v,u,v) = \frac{1}{4}\langle [U,V]_p\,, [U,V]_p\rangle \geq 0.$$

2. Codazzi equation

Let $M \subset \overline{M}$ be a submanifold of the Riemannian manifold $(\overline{M}, \overline{g})$. For the second fundamental form h of M, we define

$$(D^\perp_X h)(Y,W) := (\overline{D}_X(h(Y,W))^\perp - h(D_X Y, W) - h(Y, D_X W),$$

where $W, X, Y \in \Gamma(TM)$. Show that the Codazzi equation

$$\big(\overline{R}(X,Y)W\big)^\perp = (D^\perp_X h)(Y,W) - (D^\perp_Y h)(X,W)$$

holds for all $W, X, Y \in \Gamma(TM)$.

Solution. As $\overline{D}_Z W = D_Z W + h(Z,W)$ for $W, Z \in \Gamma(TM)$, we get

$$\begin{aligned}
\overline{R}(X,Y)W &= \overline{D}_X\overline{D}_Y W - \overline{D}_Y\overline{D}_X W - \overline{D}_{[X,Y]}W \\
&= \overline{D}_X(D_Y W + h(Y,W)) - \overline{D}_Y(D_X W + h(X,W)) \\
&\quad - (D_{[X,Y]}W + h([X,Y],W)) \\
&= D_X D_Y W + h(X, D_Y W) + \overline{D}_X(h(Y,W)) \\
&\quad - D_Y D_X W - h(Y, D_X W) - \overline{D}_Y(h(X,W)) \\
&\quad - D_{[X,Y]}W - h(D_X Y - D_Y X, W) \\
&= R(X,Y)W \\
&\quad + \overline{D}_X(h(Y,W)) - h(D_X Y, W) - h(Y, D_X W) \\
&\quad - \overline{D}_Y(h(X,W)) + h(D_Y X, W) + h(X, D_Y W).
\end{aligned}$$

Note that we used that D is torsion free, i.e. $[X,Y] = D_X Y - D_Y X$. Now, taking the normal part, we conclude that the Codazzi equation

$$\big(\overline{R}(X,Y)W\big)^\perp = (D^\perp_X h)(Y,W) - (D^\perp_Y h)(X,W)$$

holds.

3. Sectional curvature of submanifolds

Let $(\overline{M}, \overline{g})$ be a Riemannian manifold with sectional curvature $\overline{\sec}$. Let $p \in \overline{M}$ and $L \subset T\overline{M}_p$ an m-dimensional linear subspace.

a) Prove that there is some $r > 0$ such that for the open ball $B_r(0) \subset T\overline{M}_p$, the set $M := \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\overline{M}$.

b) Let g be the induced metric on M and let sec be the sectional curvature of M. Show that for $E \subset TM_p$, we have $\sec_p(E) = \overline{\sec}_p(E)$ and if L is a 2-dimensional subspace, then $\sec \leq \overline{\sec}$ on M.

Solution. a) First, we know that there is some $r > 0$ such that the restriction of the exponential map to $B_r(0)$, i.e. $\exp_p|_{B_r(0)} \colon B_r(0) \to \exp_p(B_r(0))$, is a diffeomorphism. Furthermore, note that $L \cap B_r(0)$ is an m-dimensional submanifold of $B_r(0)$ and hence $M = \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\exp_p(B_r(0))$. Finally, as $\exp_p(B_r(0))$ is open in $\overline{M}$, it follows that M is a submanifold of $\overline{M}$ as well.

b) Let $u, v \in E$ be an orthonormal basis of $E \subset TM_p$. Then we have

$$\begin{aligned}
\sec_p(E) &= R_p(u, v, u, v) \\
&= \overline{R}_p(u, v, u, v) + \overline{g}_p(h_p(u, u), h_p(v, v)) - \overline{g}_p(h_p(u, v), h_p(u, v)) \\
&= \overline{\sec}_p(E) + \overline{g}_p(h_p(u, u), h_p(v, v)) - \overline{g}_p(h_p(u, v), h_p(u, v))
\end{aligned}$$

We now prove that $h_p(u, u) = h_p(v, v) = h_p(u, v) = 0$. Extend u, v to an orthonormal basis $e_1 = u, e_2 = v, e_3, \ldots, e_{\overline{m}}$ of $T\overline{M}_p$. Then this basis induces normal coordinates on $\overline{M}$. Hence, we have $\Gamma_{ij}^k(p) = 0$ and thus $(\overline{D}_{e_i} e_j)_p = 0$ for all i, j. In particular this implies that $h_p(u, u) = h_p(v, v) = h_p(u, v) = 0$ as claimed.

Assume now that $L \subset T\overline{M}_p$ is 2-dimensional and let $q := \exp_p(x) \in M$ for $x \in L \cap B_r(0)$. By the above, we may assume that $x \neq 0$.

Define $w := \frac{x}{|x|} \in TM_p$ and let c_w be the unique geodesic with $c(0) = p$ and $\dot{c}(0) = w$. Then we have $q = c_w(|x|)$ and $u := \dot{c}_w(|x|) \in TM_q$ with $|u| = 1$. Furthermore, by Lemma 2.17,we get

$$h_q(u, u) = \left(\frac{\overline{D}}{dt} \dot{c}_w \bigg|_{t=|x|} \right)^{\perp} = 0.$$

To compute the sectional curvature of $E = TM_q$, we extend u to an orthonormal basis u, v of E and get

$$\sec_q(E) = R_q(u, v, u, v) = \overline{R}_q(u, v, u, v) - |h_q(u, v)|^2 \leq \overline{\sec}_q(E)$$

as desired.