

Solutions 7

1. Jacobi fields in space forms

Let M be a space form with curvature $\kappa \in \mathbb{R}$. Furthermore, let $c: \mathbb{R} \rightarrow M$ be a geodesic which is parametrized by arc length and $N_0 \in TM_{c(0)}$ with $|N_0| = 1$, $\langle N_0, \dot{c}(0) \rangle = 0$. Determine the Jacobi field Y along c with starting conditions $Y(0) = aN_0$ and $\dot{Y}(0) = bN_0$ for $a, b \in \mathbb{R}$.

Solution. From Exercise 1 in Sheet 5, we know that

$$R(X, Y)W = \kappa (g(Y, W)X - g(X, W)Y)$$

and thus the Jacobi equation is

$$\ddot{Y} + \kappa (g(\dot{c}, \dot{c})Y - g(Y, \dot{c})\dot{c}) = 0.$$

Consider now an orthonormal basis $\dot{c}(0), N_0, X_0^3, \dots, X_0^n$ of $TM_{c(0)}$ and the corresponding parallel vector fields $\dot{c}, N, X^3, \dots, X^n \in \Gamma(c^*TM)$. We make the Ansatz $Y = f\dot{c} + gN + \sum_{i=3}^n h_i X^i$ and get

$$\begin{aligned} \dot{Y} &= \dot{f}\dot{c} + \dot{g}N + \sum_{i=3}^n \dot{h}_i X^i, \\ \ddot{Y} &= \ddot{f}\dot{c} + \ddot{g}N + \sum_{i=3}^n \ddot{h}_i X^i. \end{aligned}$$

If we insert it into the Jacobi equation, this yields

$$\ddot{f}\dot{c} + \ddot{g}N + \sum_{i=3}^n \ddot{h}_i X^i + \kappa \left(f\dot{c} + gN + \sum_{i=3}^n h_i X^i - f\dot{c} \right) = 0,$$

which is

$$\begin{aligned} \ddot{f} &= 0, & f(0) &= 0, & \dot{f}(0) &= 0, \\ \ddot{g} + \kappa g &= 0, & g(0) &= a, & \dot{g}(0) &= b, \\ \ddot{h}_i + \kappa h_i &= 0, & h_i(0) &= 0, & \dot{h}_i(0) &= 0. \end{aligned}$$

First, we note that this implies $f \equiv 0$ and $h_i \equiv 0$. Hence, we have $Y(t) = g(t)N(t)$. Finally, we distinguish three cases to determine g :

- If $\kappa < 0$, we get $g(t) = a \cosh(\sqrt{-\kappa}t) + b \sinh(\sqrt{-\kappa}t)$,
- if $\kappa = 0$, we get $g(t) = a + bt$, and
- if $\kappa > 0$, we get $g(t) = a \cos(\sqrt{\kappa}t) + b \sin(\sqrt{\kappa}t)$.

2. Trace of a symmetric bilinear form

Let $(V, \langle \cdot, \cdot \rangle)$ be a m -dimensional Euclidean space and let $r: V \times V \rightarrow \mathbb{R}$ be a symmetric bilinear form. Furthermore, let $S^{m-1} = \{v \in V : |v| = 1\}$ be the unit sphere. Prove that

$$\int_{S^{m-1}} r(v, v) \, d\text{vol}^{S^{m-1}} = \frac{\text{vol}(S^{m-1})}{m} \text{tr}(r) = \omega_m \text{tr}(r),$$

where $d\text{vol}^{S^{m-1}}$ denotes the induced volume on S^{m-1} and ω_m is the volume of the m -dimensional unit ball.

Solution. Let e_1, \dots, e_m be a $(\langle \cdot, \cdot \rangle)$ -orthonormal basis of V such that r is diagonal, that is, $r(v, v) = \sum_{j=1}^m \lambda_j v_j^2$. Moreover, let $\tau_0, \dots, \tau_{m-1} \in \text{SO}(m, \mathbb{R})$ be the isometries defined by $\tau_i e_j := e_{i+j \bmod m}$. Then we have

$$\begin{aligned} \int_{S^{m-1}} r(v, v) \, d\text{vol}^{S^{m-1}} &= \frac{1}{m} \sum_{i=0}^{m-1} \int_{S^{m-1}} r(\tau_i v, \tau_i v) \, d\text{vol}^{S^{m-1}} \\ &= \frac{1}{m} \int_{S^{m-1}} \text{tr}(r) \, d\text{vol}^{S^{m-1}} = \frac{\text{vol}(S^{m-1})}{m} \text{tr}(r), \end{aligned}$$

since

$$\begin{aligned} \sum_{i=0}^{m-1} r(\tau_i v, \tau_i v) &= \sum_{i=0}^{m-1} r\left(\sum_{j=1}^m v_j e_{i+j}, \sum_{j=1}^m v_j e_{i+j}\right) \\ &= \sum_{j=1}^m \left(\sum_{i=0}^{m-1} \lambda_{i+j}\right) v_j^2 \\ &= \text{tr}(r) \sum_{j=1}^m v_j^2 = \text{tr}(r). \end{aligned}$$

Finally, recall that $\omega_m = \int_0^1 r^{m-1} \text{vol}(S^{m-1}) \, dr = \frac{\text{vol}(S^{m-1})}{m}$.

3. Small balls and scalar curvature

Let p be a point in the m -dimensional Riemannian manifold (M, g) . The goal is to prove the following Taylor expansion of the volume of the ball $B_r(p)$ as a function of r :

$$\text{vol}(B_r(p)) = \omega_m r^m \left(1 - \frac{1}{6(m+2)} \text{scal}(p)r^2 + \mathcal{O}(r^3) \right).$$

- a) Let $v \in TM_p$ with $|v| = 1$, define the geodesic $c(t) := \exp_p(tv)$ and let $e_1 = v, e_2, \dots, e_m \in TM_p$ be an orthonormal basis. Consider the Jacobi fields Y_i along c with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$ for $i = 2, \dots, m$. Show that the volume distortion factor of \exp_p at tv is given by

$$J(v, t) := \sqrt{\det(\langle T_{tv}e_i, T_{tv}e_j \rangle)} = t^{-(m-1)} \sqrt{\det(\langle Y_i, Y_j \rangle)},$$

where $T_{tv} := (d \exp_p)_{tv}$.

- b) Let E_2, \dots, E_m be parallel vector fields along c with $E_i(0) = e_i$. Then the Taylor expansion of Y_i is

$$Y_i(t) = tE_i - \sum_{k=2}^m \left(\frac{t^3}{6} R(e_i, v, e_k, v) + \mathcal{O}(t^4) \right) E_k.$$

- c) Conclude that $J(v, t) = 1 - \frac{t^2}{6} \text{ric}(v, v) + \mathcal{O}(t^4)$.

Hint: Use $\det(I_m + \epsilon A) = 1 + \epsilon \text{tr}(A) + \mathcal{O}(\epsilon^2)$.

- d) Prove the above formula for $\text{vol}(B_r(p))$.

Solution. a) As we have seen in the proof of Proposition 3.6, the Jacobi fields Y_i are given as variation vector fields along c of $\alpha_i(s, t) := \exp_p(t(v + se_i))$, i.e.

$$Y_i(t) = \left. \frac{d}{ds} \right|_{s=0} \alpha_i(s, t) = T_{tv}(te_i)$$

and therefore $T_{tv}e_i = \frac{1}{t}Y_i(t)$.

Furthermore, we have $\langle T_{tv}v, T_{tv}e_i \rangle = \langle v, e_i \rangle = 0$ by the Gauss Lemma. Then the volume distortion is given by

$$J(v, t) = \sqrt{\det(\langle T_{tv}e_i, T_{tv}e_j \rangle)} = t^{-(m-1)} \sqrt{\det(\langle Y_i, Y_j \rangle)}.$$

b) We check that the derivatives coincide. Clearly, we have $Y_i(0) = 0$, $\dot{Y}_i(0) = e_i$ and $\ddot{Y}_i(0) = -R(Y_i(0), \dot{c}(0))\dot{c}(0) = 0$. Furthermore,

$$\begin{aligned}\ddot{Y}_i(0) &= -(D_{\dot{c}}R)(Y_i(0), \dot{c}(0))\dot{c}(0) - R(\dot{Y}_i(0), \dot{c}(0))\dot{c}(0) \\ &= -R(e_i, v)v = -\sum_{k=2}^m \langle R(e_i, v)v, e_k \rangle e_k = -\sum_{k=2}^m R(e_k, v, e_i, v)e_k.\end{aligned}$$

c) With the above, we get

$$\begin{aligned}\langle Y_i, Y_j \rangle &= t^2 \langle E_i, E_j \rangle - \frac{t^4}{6} \sum_{k=2}^m R(e_i, v, e_k, v) \langle E_k, E_j \rangle \\ &\quad - \frac{t^4}{6} \sum_{k=2}^m R(e_j, v, e_k, v) \langle E_i, E_k \rangle + \mathcal{O}(t^5) \\ &= t^2 \delta_{ij} - \frac{t^4}{3} R(e_i, v, e_j, v) + \mathcal{O}(t^5)\end{aligned}$$

and thus

$$\begin{aligned}J(v, t) &= \sqrt{\det \left(\delta_{ij} - \frac{t^2}{3} R(e_i, v, e_j, v) + \mathcal{O}(t^3) \right)} \\ &= \sqrt{1 - \frac{t^2}{3} \operatorname{tr}(R(e_i, v, e_j, v)) + \mathcal{O}(t^3)} \\ &= 1 - \frac{t^2}{6} \operatorname{ric}(v, v) + \mathcal{O}(t^3).\end{aligned}$$

d) First, we use polar coordinates:

$$\operatorname{vol}(B_r(p)) = \int_{B_r(0)} J(v, t) \, dx^1 \dots dx^m = \int_0^r \int_{S^{m-1}} t^{m-1} J(v, t) \, d\operatorname{vol}^{S^{m-1}} \, dt.$$

Then, using exercise 2 and the above, we get

$$\begin{aligned}\operatorname{vol}(B_r(p)) &= \int_0^r \int_{S^{m-1}} t^{m-1} \left(1 - \frac{t^2}{6} \operatorname{ric}(v, v) + \mathcal{O}(t^3) \right) \, d\operatorname{vol}^{S^{m-1}} \, dt \\ &= \int_0^r t^{m-1} \left(\operatorname{vol}(S^{m-1}) - \frac{t^2}{6} \int_{S^{m-1}} \operatorname{ric}(v, v) \, d\operatorname{vol}^{S^{m-1}} + \mathcal{O}(t^3) \right) \, dt \\ &= \frac{r^m}{m} m \omega_m - \frac{r^{m+2}}{6(m+2)} \operatorname{scal}(p) \omega_m + \mathcal{O}(r^{m+3}) \\ &= \omega_m r^m \left(1 - \frac{r^2}{6(m+2)} \operatorname{scal}(p) + \mathcal{O}(r^3) \right).\end{aligned}$$