Solutions 9

1. Almost complex structure

An almost complex structure J on a manifold M^m is a $(1,1)$-tensor field with the following property: if for every $p \in M$ we denote by $J_p \colon TM_p \to TM_p$ the linear map associated with J (recall Theorem T.3), then

$$J_p \circ J_p = -\operatorname{id}_{TM_p}.$$

Prove that every complex manifold admist an almost complex structure.
Hint: Composed with the differential of a complex chart $\varphi \colon U \to \varphi(U) \subset \mathbb{C}^n$, J_p amounts to the multiplication by i.

Solution. Let $\varphi \colon U \to \varphi(U) \subset \mathbb{C}^n$ be a chart with coordinates $(x_1, y_1, \ldots, x_n, y_n)$. As suggested by the hint, we define J locally by

$$J\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial y_i}, \qquad J\Big(\frac{\partial}{\partial y_i}\Big) = -\frac{\partial}{\partial x_i},$$

for $i = 1, \ldots, n$.

It follows that $J_p \circ J_p = -\operatorname{id}_{TM_p}$. It remains to show that J is (globally) well defined. Let $\psi \colon V \to \psi(V) \subset \mathbb{C}^n$ be another complex chart on M such that $U \cap V \neq \emptyset$ and denote the coordinates on V by $(u_1, v_1, \ldots, u_n, v_n)$, then

$$\begin{aligned}
\frac{\partial}{\partial x_k} &= \sum_i \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial u_i} + \frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial v_i}, \\
\frac{\partial}{\partial y_k} &= \sum_i \frac{\partial u_i}{\partial y_k}\frac{\partial}{\partial u_i} + \frac{\partial v_i}{\partial y_k}\frac{\partial}{\partial v_i}.
\end{aligned}$$

Since $\psi \circ \varphi^{-1}$ is biholomorphic, the Cauchy-Riemann equations imply that

$$\begin{aligned}
\frac{\partial u_i}{\partial x_k} &= \frac{\partial v_i}{\partial y_k}, \\
\frac{\partial u_i}{\partial y_k} &= -\frac{\partial v_i}{\partial x_k}.
\end{aligned}$$

Denote by J' the corresponding map, defined on V with respect to ψ, then

$$\begin{aligned}
J'\big(\tfrac{\partial}{\partial x_k}\big) &= J'\Big(\sum_i \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial u_i} + \frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial v_i}\Big) \\
&= \sum_i \frac{\partial u_i}{\partial x_k}\frac{\partial}{\partial v_i} - \frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial u_i} \\
&= \sum_i \frac{\partial v_i}{\partial y_k}\frac{\partial}{\partial v_i} + \frac{\partial u_i}{\partial y_k}\frac{\partial}{\partial u_i} \\
&= \frac{\partial}{\partial y_k}
\end{aligned}$$

and similarly $J'\big(\frac{\partial}{\partial y_k}\big) = -\frac{\partial}{\partial x_k}$. This shows that J and J' coincide on $U \cap V$.

2. Kähler manifolds

Let M be a complex manifold with an almost complex structure $J \in \Gamma(T_{1,1}M)$ (as in Exercise 1). Suppose that M is endowed with an hermitian metric, that is, $g_p(J_p v, J_p w) = g_p(v, w)$ for all $p \in M$ and $v, w \in TM_p$. Show that

$$\omega(X,Y) \coloneqq g(X, JY) \quad (X, Y \in \Gamma(TM))$$

defines a 2-form $\omega \in \Omega^2(M)$, which is closed if and only if J is parallel (i.e. $DJ = D^{1,1}J \equiv 0$).

Solution. By definition ω is a $(0,2)$-tensor field in $\Gamma(T_{0,2}M)$. We still have to show that it's antisymmetric and for that we'll use that $J_p^2 = -\operatorname{id}_{TM_p}$. For $X, Y \in \Gamma(TM)$

$$\omega(X,Y) = g(X,JY) = g(JX, J^2Y) = -g(JX,Y) = -g(Y,JX) = -\omega(Y,X),$$

thus $\omega \in \Omega^2(M)$.

In order to prove the second statement, we'll prove the following two identities

$$\begin{aligned} d\omega(X,Y,Z) &= g(X,(D_Z J)Y) + g(Y,(D_X J)Z) + g(Z,(D_Y J)X) \\ 2g(D_X(JY),Z) &= d\omega(X,JY,JZ) - d\omega(X,Y,Z). \end{aligned}$$

Let X, Y, Z, JX, JY, JZ be coordinate vector fields on a chart of M, in particular they commute. Then (see Theorem 11.3 of Differential Geometry I)

$$\begin{aligned} d\omega(X,Y,Z) &= X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) \\ &= Xg(Y,JZ) - Yg(X,JZ) + Zg(X,JY) \\ &= Xg(Y,JZ) + Yg(Z,JX) + Zg(X,JY) \end{aligned}$$

Thus by the compatibility of the Levi-Civita connection with g and the product rule $D_X(JZ) = (D_X J)Z + J(D_X Z)$ for tensor derivations (similarly for the other tuples), we compute

$$\begin{aligned} d\omega(X,Y,Z) &= Xg(Y,JZ) + Yg(Z,JX) + Zg(X,JY) \\ &= g(D_XY,JZ) + g(Y,D_X(JZ)) + g(D_YZ,JX) \\ &\quad + g(Z,D_Y(JX)) + g(D_ZX,JY) + g(X,D_Z(JY)) \\ &= g(D_XY,JZ) + g(D_YZ,JX) + g(D_ZX,JY) \\ &\quad + g(Y,JD_XZ) + g(Z,JD_YX) + g(X,JD_ZY) \\ &\quad + g(Y,(D_XJ)Z) + g(Z,(D_YJ)X) + g(X,(D_ZJ)Y) \\ &= g(Y,(D_XJ)Z) + g(Z,(D_YJ)X) + g(X,(D_ZJ)Y), \end{aligned}$$

where in the last step we have used that g is hermitian and the vector fields commute. This proves the first identity.

For the second identity first note that

$$g((D_X J)Y, Z) = g(D_X(JY), Z) - g(JD_X Y, Z) = g(D_X(JY), Z) + g(D_X Y, JZ).$$

Now, by the Koszul formula we have

$$\begin{aligned} 2g(D_X(JY), Z) &= Xg(JY, Z) + JYg(X, Z) - Zg(X, JY) \\ &= X\omega(Z, Y) - JYg(X, JJZ) - Z\omega(X, Y) \\ &= -X\omega(Y, Z) - JY\omega(X, JZ) - Z\omega(X, Y) \end{aligned}$$

and

$$\begin{aligned} 2g(D_X Y, JZ) &= Xg(Y, JZ) + Yg(X, JZ) - JZg(X, Y) \\ &= -Xg(JZ, JJY) + Y\omega(X, Z) + JZg(X, JJY) \\ &= -X\omega(JZ, JY) + Y\omega(X, Z) + JZ\omega(X, JY). \end{aligned}$$

By summing the two expression we obtain the second identity:

3. Translations

Suppose that Γ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

a) Show that there exist linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \left\{ x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \dots, z_k) \in \mathbb{Z}^k \right\} \simeq \mathbb{Z}^k.$$

b) Let l denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that l equals the length of the shortest non-zero vector of the form $\sum_{i=1}^{k} z_i v_i$ with $z_i \in \mathbb{Z}$ as above.

Solution. a) For each $g \in \Gamma$ there is some $v_g \in \mathbb{R}^m$ such that $gx = x + v_g$ for all $x \in \mathbb{R}^m$ and since Γ acts freely, we have $v_g \neq 0$ for $g \neq \mathrm{id}$. We denote $V := \{v_g \in \mathbb{R}^m : g \in \Gamma\}$. Note that, as Γ acts properly discontinuously, $V \cap B_r(0)$ is finite for all $r > 0$ and thus each subset of V has an element of minimal length.

We now do induction on m. For $m = 1$, choose $g \in \Gamma \setminus \{\mathrm{id}\}$ such that $|v_g|$ is of minimal length. If there is some $v \in V$ with $v = \lambda v_g$, $\lambda \notin \mathbb{Z}$, we also have $w := v - \lfloor \lambda \rfloor v_g \in V \setminus \{0\}$ with $|w| < |v_g|$, a contradiction to minimality.

For $m \geq 2$, let $v_g \in V \setminus \{0\}$ be of minimal length and let $V' := \mathrm{span}(v_g) \cap V$. By the same argument as above, we get $V' = \mathbb{Z}v_g$.

Then we have $\mathbb{R}^m = \mathbb{R}^{m-1} \oplus \mathbb{R}v_g$ with projection map $\pi\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$ and $\Gamma' := \Gamma/g\mathbb{Z}$ acts by translations on $\mathbb{R}^{m-1}$ via $[h]x = x + \pi(v_h)$. As for $h \notin g\mathbb{Z}$ we have $\pi(v_h) \neq 0$, this action is free. We claim that it is properly discontinuous as well. If not, there are $(h_n)_{n\in\mathbb{N}} \in \Gamma$ with $\pi(v_{h_n}) \neq \pi(v_{h_{n'}})$ and $|\pi(v_{h_n})| < r$ for some $r > 0$. But then, there are $l_n \in \mathbb{Z}$ such that $|v_{h_n} - \pi(v_{h_n}) - l_n v_g| < |v_g|$, i.e. $(v_{h_n - l_n g})_{n\in N}$ is an infinite subset of $V \cap B_{r+|v_g|}(0)$, contradicting that Γ acts properly discontinuously.

By our induction hypothesis, there are $h_2, \ldots, h_k \in \Gamma$ such that

$$\pi(V) = \mathbb{Z}\pi(v_{h_2}) \oplus \ldots \oplus \mathbb{Z}\pi(v_{h_k})$$

and consequently $V = \mathbb{Z}v_g \oplus \mathbb{Z}v_{h_2} \oplus \ldots \oplus \mathbb{Z}v_{h_k}$.

b) Let $\pi\colon \mathbb{R}^m \to \mathbb{R}^m/\Gamma$ denote the covering map and let $c\colon [0,1] \to \mathbb{R}^m/\Gamma$ be a closed curve in $\mathbb{R}^m/\Gamma$. Then for $p \in \pi^{-1}(c(0))$, there exists a unique lift $\overline{c}\colon [0,1] \to \mathbb{R}^m$ of c with $\overline{c}(0) = p$. Furthermore, if c is not null-homotopic, we have $q := \overline{c}(1) \neq \overline{c}(0)$ and therefore

$$L(c) = L(\overline{c}) \geq d(p,q) = \left|\sum_{i=1}^{k} z_i v_i\right|,$$

for some $(z_1, \ldots, z_k) \in \mathbb{Z}^k \setminus \{0\}$.

Finally, if $v = \sum_{i=1}^k z_i v_i \neq 0$ is of minimal length, then $c\colon [0,1] \to \mathbb{R}^m/\Gamma$, $c(t) := \pi(tv)$, has length $L(c) = |v|$.