Exercise 1. Let $B$ be a Brownian motion started from $x_0 \in \mathbb{R}$. For $t > 0$, let $p_t$ be the density of a $N(0,t)$ random variable, i.e. $p_t(x) = (2\pi t)^{-1/2} e^{-x^2/(2t)}$ for $x \in \mathbb{R}$. Show that for $0 = t_0 < t_1 < \cdots < t_k$ and $A_1, \ldots, A_k$ measurable subsets of $\mathbb{R}$, we have
\[
P(B_{t_i} \in A_i, \ldots, B_{t_k} \in A_k)
= \int_{A_1 \times \cdots \times A_k} p_{t_1-t_0}(x_1-x_0) \cdots p_{t_k-t_{k-1}}(x_k-x_{k-1}) \, dx_1 \cdots dx_k.
\]

Exercise 2. Let $B$ be a standard Brownian motion.

(i) Show that there exists $c > 0$ such that for all $n \geq 1$,
\[
P(\sup_{[0,n]} |B| \leq 1) \leq \mathbb{P}(|B_1 - B_0|, \ldots, |B_n - B_{n-1}| \leq 2) \leq e^{-cn}
.
\]
Deduce that $\mathbb{P}(\sup_{[0,1]} |B| \leq 1/n) \leq e^{-cn^2}$ for all $n \geq 1$.

(ii) Observe that, whenever $X$ is a centred Gaussian random variable, we have that $\mathbb{P}(|X - y| \leq r) \leq \mathbb{P}(|X| \leq r)$ for $r \geq 0$ and $y \in \mathbb{R}$. Hence show that for all continuous $f: [0,1] \to \mathbb{R}$,
\[
P(\sup_{[0,1]} |B - f| \leq 1/n) \leq e^{-cn^2}.
\]

Exercise 3. Let $B$ be a standard Brownian motion. For each $n \geq 1$, let $B^{(n)}$ be the (random) function such that $B^{(n)}_t = B_t$ for all $t \in 2^{-n}\mathbb{N}_0$ and such that $B^{(n)}$ is linear on the intervals $[i2^{-n},(i+1)2^{-n}]$ for all $i \geq 0$ (these processes appeared in the dyadic construction of Brownian motion). Fix $\epsilon > 0$ and let $f: [0,1] \to \mathbb{R}$ be continuous with $f_0 = 0$.

(i) Show that $\mathbb{P}(\sup_{[0,1]} |B - B^{(n)}| \leq \epsilon/3) \to 1$ as $n \to \infty$.

(ii) Prove that for all $n \geq 0$, $B^{(n)}$ and $B - B^{(n)}$ are independent.

(iii) Using uniform continuity of $f$, establish that $\mathbb{P}(\sup_{[0,1]} |B - f| \leq \epsilon) > 0$.

Exercise 4. Let $B$ be a standard Brownian motion. We will now show that $\mathbb{E}(\sup_{[0,1]} |B|^p) < \infty$ for all $p < \infty$ (later, using the reflection principle, the tail of the random variable $\sup_{[0,1]} |B|$ will in fact be determined explicitly).

(i) Show that
\[
\sup_{[0,1]} |B| \leq \sum_{n \geq 1} \sup_{i=0,\ldots,2^{n-1}} |B_{(i+1)2^{-n}} - B_{i2^{-n}}|.
\]

(ii) For $p \geq 1$, deduce that
\[
\mathbb{E}(\sup_{[0,1]} |B|^p)^{1/p} \leq \sum_{n \geq 1} \left( \sum_{i=0,\ldots,2^{n-1}} \mathbb{E}(|B_{(i+1)2^{-n}} - B_{i2^{-n}}|^p) \right)^{1/p}.
\]

(iii) Hence deduce that $\mathbb{E}(\sup_{[0,1]} |B|^p) < \infty$ for all $p < \infty$ sufficiently large and therefore actually for all $p \in (0,\infty)$. 

BROWNIAN MOTION AND STOCHASTIC CALCULUS (D-MATH)

EXERCISE SHEET 2
Exercise 5. For a compact set $K \subset \mathbb{R}$, we define its lower Minkowski content of dimension $d > 0$ to be

$$m_d(K) = \liminf_{n \to \infty} \frac{1}{n^d} \sum_{i \in \mathbb{Z}} 1(K \cap [i/n, (i + 1)/n] \neq \emptyset) \in [0, \infty].$$

Let $B$ be a standard Brownian motion and define $K = \{ t \in [0,1]: B_t = 0 \}$. The goal of this question is to show that for $d > 1/2$, $m_d(K) = 0$ a.s. (which means that the lower Minkowski dimension of $K$ is $\leq 1/2$ a.s.).

(i) Show that $m_d(K)$ is measurable.

(ii) Prove that

$$\mathbb{E}(m_d(K)) \leq \liminf_{n \to \infty} \frac{1}{n^d} \sum_{i=0}^{n-1} \mathbb{P}(K \cap [i/n, (i + 1)/n] \neq \emptyset) \leq \liminf_{n \to \infty} \frac{1}{n^d} \sum_{i=0}^{n-1} \mathbb{P}(\sup_{[0,1/n]} |B_{i/n} - B_{i/n}| \geq |B_{i/n}|).$$

(iii) Using the scaling and the weak Markov property of Brownian motion, show that

$$\mathbb{P}(\sup_{[0,1/n]} |B_{i/n} - B_{i/n}| \geq |B_{i/n}|) = \mathbb{P}(\sup_{[0,1]} |B| \geq \sqrt{i} |N|)$$

where $N \sim N(0,1)$ is independent of $B$.

(iv) Using exercise 4 and part (iii) above, show that for all $\alpha \in (0, 1/2)$ there exists $c'_\alpha > 0$ such that whenever $i \geq 1$, we have

$$\mathbb{P}(\sup_{[0,1/n]} |B_{i/n} - B_{i/n}| \geq |B_{i/n}|) \leq c'_\alpha / i^\alpha.$$

(v) Deduce that $\mathbb{E}(m_d(K)) = 0$ and hence $m_d(K) = 0$ a.s. for $d > 1/2$.

Exercise 6. Let $X, Y \sim N(0,1)$ be independent and let

$$R = (X^2 + Y^2)/2 \quad \text{and} \quad U = Y/X \cdot 1(X \neq 0).$$

Show that $(X, Y)$ satisfies rotational invariance i.e. prove that for $\theta \in \mathbb{R}$, $(X, Y)$ and $(\cos(\theta)X - \sin(\theta)Y, \sin(\theta)X + \cos(\theta)Y)$ have the same law. Deduce that $R$ and $U$ are independent and determine the distribution of $U$. Via an explicit computation, also determine the law of $R$.

Submission of solutions. Hand in by 04/03/2020 5 p.m. into your assistant’s tray in the hallway in front of HG E 65.

<table>
<thead>
<tr>
<th>Time</th>
<th>Room</th>
<th>Assistant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friday 8 a.m. - 9 a.m.</td>
<td>HG G 26.5</td>
<td>Daniel Contreras Salinas</td>
</tr>
<tr>
<td>Friday 9 a.m. - 10 a.m.</td>
<td>HG G 26.5</td>
<td>Maximilian Nitzschner</td>
</tr>
<tr>
<td>Friday 12 p.m. - 1 p.m.</td>
<td>HG G 26.3</td>
<td>Matthis Lehmkuehler</td>
</tr>
</tbody>
</table>

Office hours. Mondays and Thursdays 12 p.m. to 1 p.m. in HG G 32.6.