BROWNIAN MOTION AND STOCHASTIC CALCULUS (D-MATH)
EXERCISE SHEET 3

Exercise 1. Let \((X_t; \ t \geq 0)\) be some collection of random variables. Fix \(\alpha \in (0, 1]\) and \(p \in [1, \infty)\). We suppose that for all \(T > 0\) there exists \(C_T > 0\) and \(\epsilon_T > 0\) such that for all \(0 \leq s < t \leq T\),

\[
\mathbb{E}\left( \frac{|X_s - X_t|}{|t - s|^{\alpha}} \right)^p \leq C_T|t - s|^{1 + \epsilon_T}.
\]

Whenever \(f : [0, \infty) \to \mathbb{R}\) is a continuous function and \(T > 0\), we define the seminorm

\[
\|f\|_{C^\alpha([0, T])} = \sup_{0 \leq s < t \leq T} \frac{|f_s - f_t|}{|s - t|^{\alpha}}.
\]

The goal of this question is to prove Kolmogorov’s continuity criterion: Under the condition above, \(X\) has a continuous modification \(X'\) and

\[
\mathbb{E}(\|X'|_{C^\alpha([0, T])}^p) < \infty \quad \text{for all } T > 0.
\]

In particular, \(X'\) is Hölder continuous with exponent \(\alpha\) almost surely.

(i) Let \(D_n = 2^{-n}N_0\) and also let \(D = \bigcup_n D_n\). Show that for \(0 \leq s < t \leq T \in \mathbb{N}\) with \(s, t \in D\) satisfying \(2^{-N} \leq |t - s| < 2^{-(N-1)}\),

\[
|X_t - X_s| \leq 2 \sum_{n \geq N} \sup_{i=0, \ldots, 2^n T - 1} |X_{i2^{-n}} - X_{(i+1)2^{-n}}|.
\]

Deduce that with \(N_T\) such that \(2^{N_T} \geq T\),

\[
H_T := \sup_{s, t \in D} \frac{|X_s - X_t|}{|s - t|^{\alpha}} \leq 2 \sum_{n \geq -N_T} 2^{n\alpha} \sup_{i=0, \ldots, 2^n T - 1} |X_{i2^{-n}} - X_{(i+1)2^{-n}}|.
\]

(ii) Show that

\[
\mathbb{E}(H_T^p)^{1/p} \leq 2 \sum_{n \geq -N_T} 2^{n\alpha} \left( \sum_{i=0}^{2^n T - 1} \mathbb{E}(|X_{i2^{-n}} - X_{(i+1)2^{-n}}|^p) \right)^{1/p} < \infty.
\]

(iii) Let \(G = \cap_{T \geq 1} \{H_T < \infty\}\). Show that on \(G\), \(X\) restricted to \(D\) extends to a continuous function on \([0, \infty)\). We let \(X'\) be the process which, on the event \(G\), is given by this extension and is defined to be (constant) 0 otherwise. Show that for each fixed \(t \geq 0\), \(X_t = X'_t\) a.s. (in other words, \(X'\) is a continuous modification of \(X\)).

(iv) Show that \(H_T = \|X'|_{C^\alpha([0, T])}\) a.s. for each \(T > 0\) and deduce that

\[
\mathbb{E}(\|X'|_{C^\alpha([0, T])}^p) < \infty \quad \text{for all } T > 0,
\]

so that \(X'\) is Hölder continuous with exponent \(\alpha\) almost surely.

Exercise 2. Let \(B\) be a standard Brownian motion. Using exercise 1, show that almost surely \(B\) is Hölder continuous with exponent \(\alpha\) for all \(\alpha \in (0, 1/2)\).
**Exercise 3.** Let $B$ be a standard Brownian motion and define $X_t = e^{-t/2}B_t$ for $t \in \mathbb{R}$ (this is called the Ornstein-Uhlenbeck process). We consider $X$ in its canonical filtration $\mathcal{F}_t = \sigma(X_s: s \leq t)$ for $t \in \mathbb{R}$.

(i) Show that $X_t \sim N(0, 1)$ for all $t \in \mathbb{R}$ and that $X$ and $X_{t_0} + \cdot = (X_{t_0} + t: t \in \mathbb{R})$ have the same law for all $t_0 \in \mathbb{R}$.

(ii) Prove that $X$ and $X_{-\cdot} = (X_{-t}: t \in \mathbb{R})$ have the same law.

(iii) Prove that the process $X$ satisfies the weak Markov property: If $f: C([0, \infty), \mathbb{R}) \to [0, \infty)$ is measurable, then for any $t_0 \in \mathbb{R}$,
\[ \mathbb{E}(f(X_{t_0} + t: t \geq 0) | \mathcal{F}_{t_0}) = \mathbb{E}(f(X_{t_0} + t: t \geq 0) | X_{t_0}). \]

**Exercise 4.** Let $B$ be a standard Brownian motion. The aim of this question will be to prove the law of the iterated logarithm which says that almost surely
\[ \limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1. \]

Let $\epsilon \in (0, 1)$ and $\theta > 0$.

(i) Let $N \sim N(0, 1)$. Show that for $x > 0$,
\[ \frac{e^{-x^2/2}}{x\sqrt{2\pi}}(1 - 1/x^2) \leq \mathbb{P}(N \geq x) \leq \frac{e^{-x^2/2}}{x\sqrt{2\pi}}. \]

(ii) Suppose that $e^\theta < (1 + \epsilon)^2$. Show using the reflection principle for Brownian motion, part (i) and the (first) Borel-Cantelli Lemma that almost surely, the event
\[ \left\{ \sup_{[0, e^{\theta(n+1)}]} B \geq (1 + \epsilon)\sqrt{2e^{\theta n} \log \log(e^{\theta n})} \right\} \]
occurs for only finitely many $n \geq 1$. By considering suitable values for $\epsilon$ and $\theta$, deduce the upper bound of the law of the iterated logarithm.

(iii) Suppose that $(1 - \epsilon)^2 \leq 1 - e^{-\theta}$. Show using part (i) of the question and the (second) Borel-Cantelli Lemma that almost surely, the event
\[ \left\{ B_{e^{\theta n}} - B_{e^{\theta(n-1)}} \geq (1 - \epsilon)\sqrt{2e^{\theta n} \log \log(e^{\theta n})} \right\} \]
occurs for infinitely many $n \geq 1$. Observe that
\[ \frac{B_{e^{\theta n}} - B_{e^{\theta(n-1)}}}{\sqrt{2e^{\theta n} \log \log(e^{\theta n})}} = -\frac{B_{e^{\theta(n-1)}}}{\sqrt{2e^{\theta(n-1)} \log \log(e^{\theta(n-1)})}} \cdot e^{-\theta/2} \cdot \sqrt{\frac{\log(\theta(n-1))}{\log(\theta n)}}. \]

Using the upper bound of the law of the iterated logarithm from part (ii) applied to $-B$ and by choosing suitable values for $\epsilon$ and $\theta$, deduce the lower bound for the law of the iterated logarithm.

**Submission of solutions.** Hand in by 11/03/2020 5 p.m. into your assistant’s tray in the hallway in front of HG E 65.

**Time** | **Room** | **Assistant**
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Friday 8 a.m. – 9 a.m. | HG G 26.5 | Daniel Contreras Salinas
Friday 9 a.m. – 10 a.m. | HG G 26.5 | Maximilian Nitzschner
Friday 12 p.m. – 1 p.m. | HG G 26.3 | Matthis Lehmkuehler

**Office hours.** Mondays and Thursdays 12 p.m. to 1 p.m. in HG G 32.6.