BROWNIAN MOTION AND STOCHASTIC CALCULUS (D-MATH)
EXERCISE SHEET 5 – SOLUTION

Exercise 1. Let $K \subset \mathbb{R}^2$ be closed and non-polar. Show that for any $x \in \mathbb{R}^2$, a two-dimensional Brownian motion started from $x$ almost surely hits $K$.

Solution. Let $B$ be a planar Brownian motion. Since $K$ is non-polar, there exists $x_0 \in \mathbb{R}^2$ such that $\mathbb{P}_{x_0} (\exists t > 0: B_t \in K) > 0$. By translating everything, without loss of generality, $x_0 = 0$. For any closed set $C$, we write $\tau_C = \inf \{ t \geq 0 : B_t \in C \}$. Then there exists $0 < r < R$ such that $\mathbb{P}_0 (B([\tau_{\partial B}, \tau_{\partial B}(0)]) \cap K \neq \emptyset) > 0$. Using a mirror coupling, one can easily deduce that there exists $\epsilon > 0$ and $0 < r' < r$ such that

$$\epsilon \leq \mathbb{P}_x (B([\tau_{\partial B}, \tau_{\partial B}(0)]) \cap K \neq \emptyset) \leq \mathbb{P}_x (\tau_K < \tau_{\partial B}(0)) \quad \forall |x| \leq r'. $$

We now define stopping times inductively as follows. Let $\tau_0 = \tau_{\partial B}(0)$. For $i \geq 0$, we set

$$\tau_i = \inf \{ t \geq \tau_{i+1} : B_t \in \partial B_R(0) \} \quad \text{and} \quad \tau_{i+1} = \inf \{ t \geq \tau_{i+1} : B_t \in \partial B_R(0) \}.$$ 

Of course, for any $y \in \mathbb{R}^2$, we know $\mathbb{P}_y (\forall i \geq 0 : \tau_i, \tilde{\tau}_i < \infty) = 1$ (by neighbourhood recurrence of Brownian motion). For any $n \geq 1$,

$$\mathbb{P}_y (\tau_K = \infty) \leq \mathbb{P}_y (\forall i, \ldots, n : B([\tau_i, \tilde{\tau}_i]) \cap K = \emptyset)$$

Applying the strong Markov property of $B$ at time $\tau_n$ (and then repeating this inductively for $\tau_{n-1}, \ldots, \tau_0$) yields

$$\mathbb{P}_y (\forall i \leq n : B([\tau_i, \tilde{\tau}_i]) \cap K = \emptyset) = \mathbb{E}_y (\mathbb{P}_y (B([\tau_i, \tilde{\tau}_i]) \cap K = \emptyset | F_{\tau_i}) ; \tau_n < \infty, \forall i \leq n-1 : B([\tau_i, \tilde{\tau}_i]) \cap K = \emptyset)$$

$$= \mathbb{E}_y (\mathbb{P}_{\tau_{n-1}} (B(0, \tau_{\partial B}(0)) \cap K = \emptyset ; \tau_n < \infty, \forall i \leq n-1 : B([\tau_i, \tilde{\tau}_i]) \cap K = \emptyset)$$

$$\leq (1 - \epsilon) \mathbb{P}_y (\forall i, \ldots, n-1 : B([\tau_i, \tilde{\tau}_i]) \cap K = \emptyset, \tau_i, \tilde{\tau}_i < \infty)$$

$$\leq \cdots \leq (1 - \epsilon)^{n+1} \to 0 \quad \text{as} \ n \to \infty.$$ 

Therefore, $\mathbb{P}_y (\tau_K = \infty) = 0$ for all $y \in \mathbb{R}^2$ which was the claim.

Exercise 2. Let $(\xi_n : n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{E} (\xi_1) = 0$ and $\mathbb{E} (\xi_1^2) < \infty$. Let $S = (S_t : t \geq 0)$ be given by $S_0 = 0$, $S_n = \sum_{i=1}^{n} \xi_i$ for $n \in \mathbb{N}$ and $S$ linear on $[i, i+1]$ for all $i \geq 0$. Also, define $S^{(n)} = (S_{nt}/\sqrt{n} : t \in [0, 1])$. We will show that the sequence $(S^{(n)})$ is tight.

(i) Let $X = (X_t : t \in [0, 1])$ be a continuous process, $p \in [1, \infty)$, $\epsilon > 0$ and $\alpha \in (0, 1]$.

Show using the argument in exercise 1 on sheet 3 that

$$\mathbb{E} \left( \left( \sup_{0 \leq s < t \leq 1} \frac{|X_s - X_t|}{|s - t|^{\alpha}} \right)^p \right) \leq \left( \sum_{n \geq 0} 2^{1-\epsilon n} \right) \cdot \sup_{0 \leq s < t \leq 1} \mathbb{E} \left( \left( \frac{|X_s - X_t|}{|s - t|^{\alpha+1/p+\epsilon}} \right)^p \right).$$

(ii) Using the Arzelà-Ascoli Theorem, show that for $M > 0$ and $\alpha \in (0, 1]$, the set

$$K_{M, \alpha} = \{ f \in C([0, 1]) : f_0 = 0, \sup_{0 \leq s < t \leq 1} |f_s - f_t|/|s - t|^{\alpha} \leq M \}$$

is compact in $C([0, 1])$ (where this space is endowed with the supremum norm).
(iii) Show that there is a constant \( C > 0 \) such that \( \mathbb{E}(S_n^4) \leq C \cdot n^2 \) for all \( n \geq 0 \). Use this together with part (i) to show that there exists \( a_0 \in (0, 1] \) such that

\[
\sup_{n \geq 1} \mathbb{E} \left( \left( \sup_{0 \leq s < t \leq 1} \frac{|S_n(t) - S_n(s)|}{|s-t|^{\alpha_0}} \right)^4 \right) < \infty.
\]

(iv) Deduce from (iii) that \( \sup_{n \geq 1} \mathbb{P}(S_n \notin K_{M,a_0}) \to 0 \) as \( M \to \infty \).

**Solution.** (i) This is precisely the result proved in the solution of exercise 1 (ii) on sheet 3 with \( T = 1, \epsilon_T = \epsilon \cdot p, N_T = 0 \) and

\[
C_T = \sup_{0 \leq s < t \leq 1} \mathbb{E} \left( \left( \frac{|X_s - X_t|}{|s-t|^{\alpha+1/p+\epsilon}} \right)^p \right).
\]

(ii) \( \{f_0: f \in K_{M,\alpha}\} = \{0\} \) is bounded, so by Arzelà-Ascoli, it suffices to show that \( K_{M,\alpha} \) is an equicontinuous family of functions. Let \( \epsilon > 0 \) and let \( \delta = (\epsilon/M)^{1/\alpha} \). Then for \( f \in K_{M,\alpha} \) and \( 0 \leq s < t \leq 1 \) with \( |s-t| < \delta \),

\[
|f_s - f_t| \leq |s-t|^\alpha \sup_{0 \leq u < v \leq 1} |f_0 - f_u|/|v - y|^\alpha < \delta^\alpha M < \epsilon.
\]

This establishes equicontinuity as required.

(iii) By expanding the product in \( \mathbb{E}(S_n^4) = \mathbb{E}((\xi_1 + \cdots + \xi_n)^4) \) and using that independence of the \( (\xi_i) \) together with \( \mathbb{E}(\xi_i) \) for all \( i \geq 1 \) yields

\[
\mathbb{E}(S_n^4) = \sum_{i=1}^{n} \mathbb{E}(\xi_i^4) + \sum_{1 \leq i < j \leq n} \binom{4}{2} \mathbb{E}(\xi_i^2) \mathbb{E}(\xi_j^2) = n\mathbb{E}(\xi_1^4) + 3n(n-1)\mathbb{E}(\xi_1^2)^2
\]

\[
\leq (\mathbb{E}(\xi_1^4) + 3\mathbb{E}(\xi_1^2)^2)n^2
\]

as required (where we used the fact that all \( \xi_i \) are identically distributed above). Now consider \( n \geq 1 \) and \( 0 \leq s < t \leq 1 \). Suppose that \( s \in [i/n, (i+1)/n] \) and \( t \in [j/n, (j+1)/n] \). Then we obtain the following elementary but tedious bounds

\[
|S_{t}^{(n)} - S_{s}^{(n)}| \leq \begin{cases} 
|\xi_i|n(t-s)/\sqrt{n} & : j = i \\
(|\xi_i| + |\xi_{i+1}|)n(t-s)/\sqrt{n} & : j = i + 1 \\
|\xi_i|/\sqrt{n} + |\xi_j|/\sqrt{n} + |S_j - S_i|/\sqrt{n} & : j > i + 1
\end{cases}
\]

\[
|S_{t}^{(n)} - S_{s}^{(n)}|^4 \leq \begin{cases} 
|\xi_i|^4n^2(t-s)^4 & : j = i \\
8(|\xi_i|^4 + |\xi_{i+1}|^4)n^2(t-s)^4 & : j = i + 1 \\
27/n^2 \cdot (\xi_i^4 + \xi_j^4 + (S_j - S_i)^4) & : j > i + 1
\end{cases}
\]

where we used the Hölder inequality with exponents 4 and 4/3 to go from the fist to the second line (in fact, this can also be obtained using elementary estimates). Using the above bound on \( \mathbb{E}(S_n^1) \) and \( S_j - S_i = S_{j-i} \), we get

\[
\mathbb{E}(|S_{t}^{(n)} - S_{s}^{(n)}|) \leq C' \cdot \begin{cases} 
n^2(t-s)^4 & : j \in \{i, i+1\} \\
(1 + (j-i)^2)/n^2 & : j > i + 1
\end{cases}
\]

for some constant \( C' > 0 \). Since \( n^2(t-s)^4 \leq 4(t-s)^2 \) for \( j \in \{i, i+1\} \), and \( 1/n^2 \leq (t-s)^2 \), \( (j-i)^2/n^2 \leq 2(t-s)^2 + 2 \cdot 1/n^2 \) for \( j > i + 1 \), we get

\[
\mathbb{E}(|S_{t}^{(n)} - S_{s}^{(n)}|^4) \leq C''(t-s)^2
\]

for some constant \( C'' > 0 \) and all \( s, t \in [0, 1] \). The result is now immediate from part (i) where we take \( p = 4, \alpha_0 = \alpha = \epsilon = 1/8 \).
(iv) By the Markov inequality, we can simply observe that
\[
\mathbb{P}(S^{(n)} \notin K_{M,\alpha_0}) = \mathbb{P}\left( \sup_{0 \leq s < t \leq 1} \left| \frac{S_s^{(n)} - S_t^{(n)}}{s-t} \right| \geq M \right) \leq \frac{1}{M^4} \mathbb{E}\left( \left( \sup_{0 \leq s < t \leq 1} \left| \frac{S_s^{(n)} - S_t^{(n)}}{s-t} \right| \right)^4 \right)
\]
Taking the supremum over \( n \geq 1 \) on both sides and letting \( M \to \infty \) yields the claim (we used part (iii) for the finiteness of the supremum on the right-hand side.)

Exercise 3. Let \( B \) be a standard Brownian motion. Let us define \( B^*_t = \sup_{[0,t]} B \). The aim of this question is to prove that \( |B| \) and \( B' = B^* - B \) have the same law. It is possible to prove this via an explicit computation based on the reflection principle. Here, we will follow a different approach based on Donsker’s theorem.

(i) Let \( S = (S_t: t \geq 0) \) be a simple random walk (on \( \mathbb{Z} \)), linearly interpolated on the segments \([i, i+1]\) for \( i \geq 0 \). We define \( S' = S^* - S \) where \( S^*_t = \sup_{0 \leq t \leq 1} S \) for \( t \geq 0 \). Using Donsker’s theorem, show that \( (S'_n/\sqrt{n}: t \in [0,1]) \) converges weakly to \((B'_t: t \in [0,1])\) as \( n \to \infty \). Also show that \((|S'_n|/\sqrt{n}: t \in [0,1]) \) converges weakly to \((|B'_t|: t \in [0,1])\) as \( n \to \infty \).

(ii) Let \( (\epsilon_k) \) be a sequence of i.i.d. \( U([\pm 1]) \) random variables, independent of \( S \). We now define \( \tilde{S} = (\tilde{S}_n: n \geq 0) \) as follows: Let \( \tilde{S}_0 = 0 \). If \( n \geq 1 \), we inductively set
\[
\tilde{S}_n = \tilde{S}_{n-1} + (S_n - S_{n-1}) - \epsilon_n 1(\tilde{S}_{n-1} = 0, S_n - S_{n-1} = \epsilon_n)
\]
Show that \( |\tilde{S}| \) and \( (S'_n: n \geq 0) \) have the same law. Observe that \( \tilde{S} \) is a discrete-time Markov chain on \( \mathbb{Z} \) and write down the transition probabilities. Use this to show that \( \mathbb{E}((S_n - \tilde{S}_n)^2)/n \to 0 \) as \( n \to \infty \).

(iii) By combining the results from (i) and (ii), show that for \( 0 \leq t_1 < \cdots < t_m \leq 1 \), \((|B_{t_1}|, \ldots, |B_{t_m}|) = d (B'_{t_1}, \ldots, B'_{t_m})\). Deduce that \( |B| \) and \( B' \) have the same law.

Solution. (i) By Donsker’s theorem, we know that \((S_{nt}/\sqrt{n}: t \in [0,1])\) converges weakly (in \( C([0,1]) \)) to \((B_t: t \in [0,1])\) as \( n \to \infty \). We define \( \Phi, \Psi: C([0,1]) \to C([0,1]) \) by
\[
\Phi(f) = |f| \quad \text{and} \quad \Psi(f) = \left( \sup_{[0,1]} |f - f_t|: t \in [0,1] \right).
\]
It is clear that \( \Phi \) and \( \Psi \) are continuous w.r.t. the supremum norm, and hence \( \Phi(S_{nt}/\sqrt{n}: t \in [0,1]) \) converges weakly to \( \Phi(B_t: t \in [0,1]) \) as \( n \to \infty \), and \( \Psi(S_{nt}/\sqrt{n}: t \in [0,1]) \) converges weakly to \( \Psi(B_t: t \in [0,1]) \) as \( n \to \infty \). This is precisely the result we were asked to prove.

(ii) Let us list the processes we are considering (here, we write \( S_n - S_{n-1} = \xi_n \) which forms a sequence of i.i.d. \( U([\pm 1]) \) random variables)

- \( S \) is a Markov chain since whenever \( n \geq 1 \), \( S_n - S_{n-1} \) is a function of \( \xi_n \) which is independent of \( S_0, \ldots, S_{n-1} \). The transition probabilities are given by \( p_{i,i\pm 1} = 1/2 \) where \( i \in \mathbb{Z} \).
- \( |S| \) is a Markov chain since for \( n \geq 1 \), \( |S_n| - |S_{n-1}| \) is a function of \( |S_{n-1}| \) and \( \xi_n \), the latter being independent of \( |S_0|, \ldots, |S_{n-1}| \). The transition probabilities are \( p_{i,i\pm 1} = 1/2 \) for \( i \geq 1 \) and \( p_{0,0} = 1 \).
- \( S' \) is a Markov chain since whenever \( n \geq 1 \), \( S'_n - S'_{n-1} \) is a function of \( S'_{n-1} \) and \( \xi_n \), with \( \xi_n \) of course independent of \( S'_{0}, \ldots, S'_{n-1} \). The transition probabilities are \( p_{i,i\pm 1} = 1/2 \) for \( i \geq 1 \) and \( p_{0,0} = p_{0,1} = 1/2 \).
Finally, we have that $\tilde{S}$ is a Markov chain since for all $n \geq 1$, $\tilde{S}_n - \tilde{S}_{n-1}$ is a function of $\tilde{S}_{n-1}$ and $(\xi_n, \epsilon_n)$ (the latter being independent of $\tilde{S}_0, \ldots, \tilde{S}_{n-1}$). The transition probabilities are $p_{i,i+1} = 1/2$ for $i \neq 0$, $p_{0,0} = 1/2$ and $p_{0,1} = 1/4$.

From this, it is clear that $|\tilde{S}|$ has the same law as $S'$. Moreover, we can compute (the inequality will be explained below)

$$\frac{1}{n} \mathbb{E}((\tilde{S}_n - S_n)^2) = \frac{1}{n} \mathbb{E}\left(\left(\sum_{i=1}^{n} \epsilon_i 1(\tilde{S}_{i-1} = 0, S_i - S_{i-1} = \epsilon_i)\right)^2\right)$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{P}(\tilde{S}_i = 0)/2 = \frac{1}{2n} \mathbb{E}\left(\sum_{i=0}^{n-1} 1(\tilde{S}_i = 0)\right)$$

$$\leq \frac{1}{n} \mathbb{E}\left(\sum_{i=0}^{n-1} 1(S_i = 0)\right) = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{P}(S_i = 0)$$

$$= \frac{1}{n} \sum_{i \geq 0, 2i \leq n-1} \binom{2i}{i} 2^{-2i}.$$  

This tends to 0 as can be seen using Stirling’s approximation. The inequality follows from the observation that we can construct the law of $\tilde{S}$ from $S$ in another way than in the definition given. We can fix i.i.d. geometrically distributed random variables $\tau_i$ with parameter 1/2 (and therefore mean 2) and let $T$ be given by

$$T_k = S_i \quad \text{for } r \in \left[ i + \sum_{j \leq i-1} (\tau_j - 1)1(S_j = 0), i + \sum_{j \leq i} (\tau_j - 1)1(S_j = 0) \right].$$

This corresponds to adding in geometric waiting times whenever $S$ hits 0 to obtain $T$. Then $T$ and $\tilde{S}$ have the same law, and moreover $\sum_{i=0}^{n-1} 1(T_i = 0) \leq \sum_{i=0}^{n-1} 1(S_i = 0)$ which yields the desired inequality after taking expectations.

(iii) Suppose that $f: \mathbb{R}^m \to \mathbb{R}$ is smooth with compact support, then

$$|\mathbb{E}(f(|B_t|: i = 1, \ldots, m)) - \mathbb{E}(f(B'_t: i = 1, \ldots, m))|$$

$$\leq |\mathbb{E}(f(|B_t|: i = 1, \ldots, m)) - \mathbb{E}(f(|S_{nt}/\sqrt{n}|: i = 1, \ldots, m))|$$

$$+ |\mathbb{E}(f(B'_t: i = 1, \ldots, m)) - \mathbb{E}(f(S'_{nt}/\sqrt{n}: i = 1, \ldots, m))|$$

$$+ |\mathbb{E}(f(|S_{nt}/\sqrt{n}|: i = 1, \ldots, m)) - \mathbb{E}(f(S'_{nt}/\sqrt{n}: i = 1, \ldots, m))|$$

The first two terms tend to 0 as $n \to \infty$ by part (i). The third term can be bounded by the following expression ($C$ being the supremum norm of the derivative of $f$)

$$C \sum_{i=1}^{m} \mathbb{E}(|S_{nt} - \tilde{S}_{nt}|/\sqrt{n}) \leq C \sum_{i=1}^{m} \mathbb{E}(|S_{nt} - \tilde{S}_{nt}|^2)^{1/2} \sqrt{n}$$

which also tends to 0 as $n \to \infty$ by part (ii) (the linear interpolation is easy to handle). By standard measure theoretic arguments, this implies the first claim. Finally, by Brownian scaling, the equality in law extends to all $0 \leq t_1 < \cdots < t_m$ and therefore $B^* - B$ and $|B|$ have the same law.
Exercise 4. Let $B$ be a standard Brownian motion and define $B_t^* = \sup_{[0,t]} B$ for $t \geq 0$. In this question we will show that the (lower) Minkowski content of dimension $d > 0$ (see sheet 2 exercise 5 for the definition) of either of the two sets

$$K = \{ t \in [0,1]: B_t = 0 \} \quad \text{and} \quad K' = \{ t \in [0,1]: B_t^* = B_t \}$$

is $\infty$ a.s. if $d < 1/2$ and 0 a.s. if $d > 1/2$ (this means that the lower Minkowski dimensions of $K$ and $K'$ are both $1/2$ a.s.).

(i) Using the equality in law of $|B|$ and $B^* - B$, deduce that for $d > 0$, $m_d(K)$ and $m_d(K')$ have the same law.

(ii) Using exercise 5 on sheet 2, show that $m_d(K), m_d(K') = 0$ a.s. for $d > 1/2$.

(iii) Fix $\alpha \in (0,1/2)$, $n \geq 1$ and $0 \leq i \leq n - 1$. By considering the cases $[i/n, (i+1)/n] \cap K' \neq \emptyset$ and $[i/n, (i+1)/n] \cap K = \emptyset$ separately, show that

$$B^*_{(i+1)/n} - B^*_{i/n} \leq H_\alpha n^{-\alpha} 1([i/n, (i+1)/n] \cap K' \neq \emptyset)$$

where $H_\alpha = \sup_{0 \leq s < t \leq 1} |B_s - B_t|/|t-s|^\alpha \in (0,\infty)$ a.s. by Hölder continuity of $B$ with exponent $\alpha$, see exercise 2 on sheet 3.

(iv) Let $d \in (0,1/2)$. Using (iii) with a suitable $\alpha \in (0,1/2)$ and the definition of the (lower) Minkowski content, show that $m_d(K') = \infty$ a.s..

(v) Deduce that $m_d(K) = \infty$ a.s. for $d \in (0,1/2)$.

Solution. (i) We fix $d > 0$. It is easy to see that the map $\Phi: C([0,1]) \rightarrow [0,\infty]$ given by $\Phi(f) = m_d([t \in [0,1]: f_t = 0])$ is measurable. The result then follows from

$$m_d(K) = \Phi(|B_t|: t \in [0,1]), \quad m_d(K') = \Phi(B^*_t - B_t: t \in [0,1])$$

and the equality in law of $(|B_t|: t \in [0,1])$ and $(B^*_t - B_t: t \in [0,1])$.

(ii) For $d > 1/2$, we established in exercise 5 on sheet 2 that $\mathbb{P}(m_d(K) = 0) = 1$. The result is then immediate as by (i), $\mathbb{P}(m_d(K') = 0) = \mathbb{P}(m_d(K) = 0)$.

(iii) Suppose that $\alpha \in (0,1/2)$ and $0 \leq i \leq n - 1$. We can always estimate

$$B^*_{(i+1)/n} - B^*_{i/n} \leq \sup_{i/n \leq s < t \leq (i+1)/n} |B_s - B_t| \leq H_\alpha n^{-\alpha}.$$

In particular, we get the claimed inequality in the $[i/n, (i+1)/n] \cap K' \neq \emptyset$ case. Now suppose that $[i/n, (i+1)/n] \cap K' = \emptyset$, then $B^*_{i/n} = B^*_{(i+1)/n}$; indeed, if we had $B^*_{i/n} < B^*_{(i+1)/n}$, then the minimal $t \in [i/n, (i+1)/n]$ with $B_t = B^*_{i/n}$ would satisfy $B^*_t - B_t = 0$.

(iv) Let us consider $d \in (0,1/2)$. Then by the definition of the lower Minkowski content, we obtain the following inequality

$$m_d(K') = \liminf_{n \rightarrow \infty} \frac{1}{n^d} \sum_{i=0}^{n-1} 1([i/n, (i+1)/n] \cap K' \neq \emptyset) \geq \liminf_{n \rightarrow \infty} \frac{1}{n^d} \sum_{i=0}^{n-1} \frac{B^*_{(i+1)/n} - B^*_{i/n}}{(1 + H_\alpha)n^{-\alpha}}$$

$$= \liminf_{n \rightarrow \infty} \frac{n^{\alpha-d} B^*_{i/n}}{1 + H_\alpha}.$$

So by taking $\alpha \in (d,1/2)$, we see that $m_d(K') = \infty$ a.s..

(v) If $d \in (0,1/2)$, then by (i) and then (iv), $\mathbb{P}(m_d(K) = \infty) = \mathbb{P}(m_d(K') = \infty) = 1$. 

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Exercise 5. Let $B$ be a standard Brownian motion and $X$ a random variable satisfying $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) < \infty$. We will show that there exists a random time $T$ with $\mathbb{E}(T) = \mathbb{E}(X^2)$ such that $B_T \mathbb{1}(T < \infty)$ and $X$ have the same law (this coupling can be used to show Donsker’s theorem when the increments of the discrete-time random walk have i.i.d. increments having the same law as $X$), this is called a Skorohod embedding. The $X = 0$ a.s. case is trivial, so we will exclude it.

(i) Given $a < 0 \leq b$, let $\tau_{a,b} = \inf\{t \geq 0 : B_t \in \{a, b\}\}$. Using optional stopping applied to the martingales (w.r.t. the canonical filtration associated to $B$) $B$ and $(B_t^2 - t : t \geq 0)$, compute $\mathbb{E}(\tau_{a,b})$, $\mathbb{P}(\tau_{a,b} = a, \tau_{a,b} < \infty)$ and $\mathbb{P}(\tau_{a,b} = b, \tau_{a,b} < \infty)$.

(ii) Let $\mu$ be the law of $X$ and define the probability measure
\[
\nu(dl_-, dl_+) = \frac{1}{Z} (l_+ - l_-) 1(l_- < 0 \leq l_+) \mu(dl_-) \mu(dl_+)
\]
(where $Z$ is a normalisation constant) and let $(L_-, L_+)$ be sampled independently of $B$ according to the distribution $\nu$. Define $T = \tau_{L_- L_+}$. By conditioning on $(L_-, L_+)$ and then using (i), verify that $\mathbb{E}(T) = \mathbb{E}(X^2)$, and that $B_T \mathbb{1}(T < \infty)$ and $X$ have the same law.

Solution. (i) Since $\sup_{[0,\infty)} B = \infty$ a.s., we know that $\tau_{a,b} < \infty$ a.s.. Let $\mathcal{F}$ be the canonical filtration associated to $B$ i.e. $\mathcal{F}_t = \sigma(B_s : s \leq t)$. Then $B$ and $M$ given by $M_t = B_t^2 - t$ are continuous martingales w.r.t. $\mathcal{F}$ as is easily verified. Therefore, by optional stopping, for $t \geq 0$, we get
\[
\mathbb{E}(\tau_{a,b} \wedge t) = \mathbb{E}(B_0) = 0 \quad \text{and} \quad \mathbb{E}(M_{\tau_{a,b} \wedge t}) = \mathbb{E}(M_0) = 0.
\]
Since $|M_t| \leq (a) \vee b$, by dominated convergence, letting $t \to \infty$, we obtain
\[
0 = \mathbb{E}(\tau_{a,b} \wedge \mathbb{1}(\tau_{a,b} < \infty)) = a \cdot \mathbb{P}(\tau_{a,b} = a, \tau_{a,b} < \infty) + b \cdot \mathbb{P}(\tau_{a,b} = b, \tau_{a,b} < \infty).
\]
The observation $\mathbb{P}(\tau_{a,b} = a, \tau_{a,b} < \infty) = 1 - \mathbb{P}(\tau_{a,b} = b, \tau_{a,b} < \infty)$ then implies that
\[
\mathbb{P}(\tau_{a,b} = a, \tau_{a,b} < \infty) = \frac{b}{b-a} \quad \text{and} \quad \mathbb{P}(\tau_{a,b} = b, \tau_{a,b} < \infty) = \frac{-a}{b-a}.
\]
Now, let us rewrite $\mathbb{E}(M_{\tau_{a,b} \wedge t}) = 0$ as $\mathbb{E}(B_{\tau_{a,b} \wedge t}^2) = \mathbb{E}(\tau_{a,b} \wedge t)$. By taking $t \to \infty$, this implies
\[
\mathbb{E}(B_{\tau_{a,b}}^2) = \mathbb{E}(\tau_{a,b} < \infty) = \mathbb{E}(\tau_{a,b}).
\]
Here, we used dominated convergence as above for the first term and monotone convergence for the second term. Putting everything together yields
\[
\mathbb{E}(\tau_{a,b}) = \mathbb{E}(B_{\tau_{a,b}}^2) = \mathbb{E}(\tau_{a,b} < \infty) = a^2 \frac{b}{b-a} + b^2 \frac{-a}{b-a} = -ab.
\]
(ii) Let us begin by showing that $E(T) = E(X^2)$. By conditioning on $(L_-, L_+)$, we get
\[
E(T) = E(\tau_{L_-, L_+}) = \int \nu(dl_-, dl_+) E(\tau_{l_-, l_+})
\]
\[
= \int \frac{1}{Z} (l_+ - l_-) 1(l_- < 0 \leq l_+) \mu(dl_-) \mu(dl_+).\]
\[
= E(X^2; X \geq 0) \int \frac{-l_1(l_- < 0)}{Z} \mu(dl_-) + E(X^2; X < 0) \int \frac{l_1(l_+ \geq 0)}{Z} \mu(dl_+).\]

Since $E(X) = 0$, the two integrals in the line above are equal; moreover, they both equal 1 by the definition of $Z$ (as a normalisation constant). Therefore, $E(T) = E(X^2)$. Now, consider a measurable function $f: \mathbb{R} \to [0, \infty)$. Then, again by first conditioning on $(L_-, L_+)$, we get
\[
E(f(B_T 1(T < \infty))) = \int \nu(dl_-, dl_+) E(f(B_{\tau_{l_-, l_+}}); \tau_{l_-, l_+} < \infty)
\]
\[
= \int \frac{1}{Z} (l_+ - l_-) 1(l_- < 0 \leq l_+) \mu(dl_-) \mu(dl_+). \frac{l_1f(l_-) - l_-f(l_+)}{l_+ - l_-}
\]
\[
= E(f(X); X < 0) \int \frac{l_1(l_+ \geq 0)}{Z} \mu(dl_+)
\]
\[
+ E(f(X); X \geq 0) \int \frac{-l_1(l_- < 0)}{Z} \mu(dl_-)
\]
\[
= E(f(X)).\]

Therefore, $B_T 1(T < \infty)$ and $X$ have the same law.