Exercise 1. Let $K \subset \mathbb{R}^2$ be closed and non-polar. Show that for any $x \in \mathbb{R}^2$, a two-dimensional Brownian motion started from $x$ almost surely hits $K$.

Exercise 2. Let $(\xi_n : n \geq 1)$ be a sequence of i.i.d. random variables such that $\mathbb{E}(\xi_1) = 0$ and $\mathbb{E}(\xi_1^2) < \infty$. Let $S = (S_t : t \geq 0)$ be given by $S_0 = 0$, $S_n = \sum_{i=1}^n \xi_i$ for $n \in \mathbb{N}$ and $S$ linear on $[i, i+1]$ for all $i \geq 0$. Also, define $S^{(n)} = (S_{nt}/\sqrt{n} : t \in [0, 1])$. We will show that the sequence $(S^{(n)})$ is tight.

(i) Let $X = (X_t : t \in [0, 1])$ be a continuous process, $p \in [1, \infty)$, $\epsilon > 0$ and $\alpha \in (0, 1]$. Show using the argument in exercise 1 on sheet 3 that

$$\mathbb{E}\left(\left(\sup_{0 \leq s \leq t \leq 1} \frac{|X_s - X_t|}{|s-t|^\alpha}\right)^p\right) \leq \left(\sum_{n \geq 0} 2^{1-\epsilon n}\right)^p \cdot \sup_{0 \leq s \leq t \leq 1} \mathbb{E}\left(\left(\frac{|X_s - X_t|}{|s-t|^{(1/\alpha)+1/p+\epsilon}}\right)^p\right).$$

(ii) Using the Arzelà-Ascoli Theorem, show that for $M > 0$ and $\alpha \in (0, 1]$, the set

$$K_{M,\alpha} = \{ f \in C([0, 1]) : f_0 = 0, \sup_{0 \leq s < t \leq 1} |f_s - f_t|/|s-t|^\alpha \leq M \}$$

is compact in $C([0, 1])$ (where this space is endowed with the supremum norm).

(iii) Show that there is a constant $C > 0$ such that $\mathbb{E}(S^{(n)}_4) \leq C \cdot n^2$ for all $n \geq 0$. Use this together with part (i) to show that there exists $a_0 \in (0, 1]$ such that

$$\sup_{n \geq 1} \mathbb{E}\left(\left(\sup_{0 \leq s < t \leq 1} \frac{|S^{(n)}_s - S^{(n)}_t|}{|s-t|^{a_0}}\right)^4\right) < \infty.$$ 

(iv) Deduce from (iii) that $\sup_{n \geq 1} \mathbb{P}(S^{(n)} \notin K_{M,a_0}) \to 0$ as $M \to \infty$.

Exercise 3. Let $B$ be a standard Brownian motion. Let us define $B^*_t = \sup_{[0,t]} B$. The aim of this question is to prove that $|B|$ and $B' = B^* - B$ have the same law. It is possible to prove this via an explicit computation based on the reflection principle. Here, we will follow a different approach based on Donsker’s theorem.

(i) Let $S = (S_t : t \geq 0)$ be a simple random walk (on $\mathbb{Z}$), linearly interpolated on the segments $[i, i+1]$ for $i \geq 0$. We define $S' = S^* - S$ where $S^*_t = \sup_{0 \leq s \leq t} S$ for $t \geq 0$. Using Donsker’s theorem, show that $(S_{nt}/\sqrt{n} : t \in [0, 1])$ converges weakly to $(B'_t : t \in [0, 1])$ as $n \to \infty$. Also show that $(|S_{nt}/\sqrt{n} : t \in [0, 1])$ converges weakly to $(|B_t| : t \in [0, 1])$ as $n \to \infty$.

(ii) Let $(\epsilon_k)$ be a sequence of i.i.d. $U(\{\pm 1\})$ random variables, independent of $S$. We now define $\tilde{S} = (\tilde{S}_n : n \geq 0)$ as follows: Let $\tilde{S}_0 = 0$. If $n \geq 1$, we inductively set

$$\tilde{S}_n = \tilde{S}_{n-1} + (S_n - S_{n-1}) - \epsilon_n 1(\tilde{S}_{n-1} = 0, S_n - S_{n-1} = \epsilon_n)$$

Show that $|\tilde{S}|$ and $(S'_n : n \geq 0)$ have the same law. Observe that $\tilde{S}$ is a discrete-time Markov chain on $\mathbb{Z}$ and write down the transition probabilities. Use this to show that $\mathbb{E}((S_n - \tilde{S}_n)^2)/n \to 0$ as $n \to \infty$.

(iii) By combining the results from (i) and (ii), show that for $0 \leq t_1 < \cdots < t_m \leq 1$, $(|B_{t_1}|, \ldots, |B_{t_m}|) = d (B_{t_1}', \ldots, B_{t_m}')$. Deduce that $|B|$ and $B'$ have the same law.
Exercise 4. Let $B$ be a standard Brownian motion and define $B^*_t = \sup_{[0,t]} B$ for $t \geq 0$. In this question we will show that the (lower) Minkowski content of dimension $d > 0$ (see sheet 2 exercise 5 for the definition) of either of the two sets

$$K = \{ t \in [0,1]: B_t = 0 \} \quad \text{and} \quad K' = \{ t \in [0,1]: B^*_t = B_t \}$$

is $\infty$ a.s. if $d < 1/2$ and 0 a.s. if $d > 1/2$ (this means that the lower Minkowski dimensions of $K$ and $K'$ are both $1/2$ a.s.).

(i) Using the equality in law of $|B|$ and $B^* - B$, deduce that for $d > 0$, $m_d(K)$ and $m_d(K')$ have the same law.

(ii) Using exercise 5 on sheet 2, show that $m_d(K), m_d(K') = 0$ a.s. for $d > 1/2$.

(iii) Fix $\alpha \in (0,1/2)$, $n \geq 1$ and $0 < i \leq n-1$. By considering the cases $[i/n, (i+1)/n] \cap K' \neq \emptyset$ and $[i/n, (i+1)/n] \cap K' = \emptyset$ separately, show that

$$B^*_{(i+1)/n} - B^*_i \leq H_\alpha n^{\alpha} \mathbb{1}([i/n, (i+1)/n] \cap K' \neq \emptyset)$$

where $H_\alpha = \sup_{0 \leq s < t \leq 1} |B_s - B_t|/|t-s|^\alpha \in (0, \infty)$ a.s. by Hölder continuity of $B$ with exponent $\alpha$, see exercise 2 on sheet 3.

(iv) Let $d \in (0,1/2)$. Using (iii) with a suitable $\alpha \in (0,1/2)$ and the definition of the (lower) Minkowski content, show that $m_d(K') = \infty$ a.s..

(v) Deduce that $m_d(K) = \infty$ a.s. for $d \in (0,1/2)$.

Exercise 5. Let $B$ be a standard Brownian motion and $X$ a random variable satisfying $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) < \infty$. We will show that there exists a random time $T$ with $\mathbb{E}(T) = \mathbb{E}(X^2)$ such that $B_T 1(T < \infty)$ and $X$ have the same law (this coupling can be used to show Donsker’s theorem when the increments of the discrete-time random walk have i.i.d. increments having the same law as $X$), this is called a Skorohod embedding. The $X = 0$ a.s. case is trivial, so we will exclude it.

(i) Given $a < 0 \leq b$, let $\tau_{a,b} = \inf \{ t \geq 0: B_t \in \{a,b\} \}$. Using optional stopping applied to the martingales (w.r.t. the canonical filtration associated to $B$) $B$ and $(B^2_t - t: t \geq 0)$, compute $\mathbb{E}(\tau_{a,b}), \mathbb{P}(B_{\tau_{a,b}} = a, \tau_{a,b} < \infty)$ and $\mathbb{P}(B_{\tau_{a,b}} = b, \tau_{a,b} < \infty)$.

(ii) Let $\mu$ be the law of $X$ and define the probability measure

$$\nu(dl_-, dl_+) = \frac{1}{Z} (l_+ - l_-) \mathbb{1}(l_- < 0 \leq l_+) \mu(dl_-) \mu(dl_+)$$

(where $Z$ is a normalisation constant) and let $(L_-, L_+)$ be sampled independently of $B$ according to the distribution $\nu$. Define $T = \tau_{L_-, L_+}$. By conditioning on $(L_-, L_+)$ and then using (i), verify that $\mathbb{E}(T) = \mathbb{E}(X^2)$, and that $B_T 1(T < \infty)$ and $X$ have the same law.

Submission of solutions. Submit by 27/03/2020 5 p.m. online to your assistant via

https://sam-up.math.ethz.ch/upload.html

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