Appendix C: Some martingale results in discrete time

This section contains a number of results on martingales and stochastic integrals in discrete time. We formulate them for a general probability space, but point out that any conditions like integrability or boundedness are trivially satisfied whenever the underlying space \( \Omega \) is finite and the time horizon is finite as well.

We start with a probability space \((\Omega, \mathcal{F}, P)\) and a filtration in discrete time given by \( \mathcal{F} = (\mathcal{F}_k)_{k=0,1,2,...} = (\mathcal{F}_k)_{k \in \mathbb{N}_0} \). We sometimes assume that \( \mathcal{F}_0 \) is \( P \)-trivial, but this is not needed in general. We also sometimes look at processes indexed only by \( k = 0,1,\ldots,T \) for some \( T \in \mathbb{N} \).

**Definition.** A stochastic process \( M = (M_k)_{k \in \mathbb{N}_0} \) is called a martingale (with respect to \( P \) and \( \mathcal{F} \)) if

1. \( M \) is adapted to \( \mathcal{F} \), meaning that \( M_k \) is \( \mathcal{F}_k \)-measurable for all \( k \);
2. \( M \) is \( P \)-integrable, meaning that \( E[|M_k|] < \infty \) or \( M_k \in L^1(P) \), for all \( k \);
3. \( M \) satisfies the martingale property that \( E[M_\ell | \mathcal{F}_k] = M_k \) \( P \)-a.s. for all \( k \leq \ell \).

If instead of (M3) we have

\[
(M^*3) \quad E[M_\ell | \mathcal{F}_k] \leq M_k \quad \text{\( P \)-a.s. for all } k \leq \ell,
\]

then \( M \) is called a supermartingale; if instead of (M3) we have

\[
(M_\ast 3) \quad E[M_\ell | \mathcal{F}_k] \geq M_k \quad \text{\( P \)-a.s. for all } k \leq \ell,
\]

then \( M \) is called a submartingale.

**Remarks.** 1) The martingale property (M3) is equivalent to

\[ E[\Delta M_k | \mathcal{F}_{k-1}] = 0 \quad \text{\( P \)-a.s. for all } k \in \mathbb{N}. \]
If we only look at a martingale $M = (M_k)_{k=0,1,...,T}$ in finite discrete time, then (M3) is also equivalent to

$$E[M_T | \mathcal{F}_k] = M_k \quad \text{P-a.s. for all } k = 0, 1, \ldots, T.$$ 

For sub- and supermartingales, the first equivalence also holds, with “=” of course replaced by “≥” and “≤”, respectively. However, the second equivalence is specific to the martingale case.

2) An analogous definition can be used for an $\mathbb{R}^m$-valued process by simply imposing the conditions on each coordinate.

Example. A first standard example for a martingale is given by successive partial sums of independent centered random variables. Suppose that $(Y_j)_{j \in \mathbb{N}}$ are independent and integrable random variables with $E[Y_j] \equiv 0$ and define $M_k := \sum_{j=1}^{k} Y_j$ (with $M_0 = 0$ by the usual convention that an empty sum is zero) as well as $\mathcal{F}_k := \sigma(Y_1, \ldots, Y_k) = \sigma(M_0, M_1, \ldots, M_k)$ for $k \in \mathbb{N}_0$. Then $M$ is a martingale with respect to $P$ and $\mathcal{F}$; this follows immediately because $\Delta M_k = Y_k$ is independent of $\mathcal{F}_{k-1}$. In complete analogy, $N_k := \prod_{j=1}^{k} R_j$ is a martingale (with $N_0 = 1$) if $(R_j)_{j \in \mathbb{N}}$ are independent and integrable with $E[R_j] \equiv 1$.

Example. A second standard example is given by successive predictions. Suppose we are given a filtration $\mathcal{F}$, let $Y$ be an integrable random variable and define $M_k := E[Y | \mathcal{F}_k]$ for $k \in \mathbb{N}_0$. Using the projectivity of conditional expectations then easily shows that $M$ is a martingale (with respect to $P$ and $\mathcal{F}$).

Martingales form a large class of stochastic processes and have many important and useful properties. Our first result shows that a stochastic integral with respect to a martingale is again a martingale if the integrand is sufficiently integrable.

**Proposition C.1.** Suppose $M$ is an $\mathbb{R}^m$-valued martingale and $H = (H_k)_{k \in \mathbb{N}}$ an $\mathbb{R}^m$-valued bounded predictable process. Then the stochastic integral $H \cdot M = \int H \, dM$ is again a
Proof. It is clear that $\int H \, dM$ is adapted and also integrable, because $H$ is bounded. For each $k$ and each coordinate $i$, $H_k^i$ is bounded and $\mathcal{F}_{k-1}$-measurable; so we have

$$E[H_k^i \Delta M_k^i \mid \mathcal{F}_{k-1}] = H_k^i E[\Delta M_k^i \mid \mathcal{F}_{k-1}] = 0 \quad \text{P-a.s.}$$

because $M$ is a martingale, and this implies that P-a.s.,

$$E\left[\Delta \left( \int H \, dM \right)_k \mid \mathcal{F}_{k-1}\right] = E[H_k \cdot \Delta M_k \mid \mathcal{F}_{k-1}] = \sum_{i=1}^k E[H_k^i \Delta M_k^i \mid \mathcal{F}_{k-1}] = 0.$$

So $\int H \, dM$ is indeed a martingale as well. \textbf{q.e.d.}

Remark. An extension of Proposition C.1 from martingales to sub- or supermartingales is not true in general. However, there is one important exception: If $n = 1$ and $M$ is a sub- or supermartingale, then $\int H \, dM$ is again a sub- or supermartingale, respectively, if $H$ is bounded, predictable and in addition nonnegative. (The proof is left as an exercise.)

Definition. If $\tau$ is a stopping time (with respect to $\mathcal{F}$), we call

$$\mathcal{F}_\tau := \{ A \in \mathcal{F} \mid A \cap \{ \tau \leq k \} \subseteq \mathcal{F}_k \text{ for all } k \in \mathbb{N}_0 \}$$

the $\sigma$-field of \textit{events observable up to time} $\tau$.

Remarks. 1) One can (and should) check easily that $\mathcal{F}_\tau$ is indeed a $\sigma$-field. One can and should also check that if $\tau \equiv k_0$, then $\mathcal{F}_\tau = \mathcal{F}_{k_0}$ so that there is no abuse of notation.

2) If $\sigma$ and $\tau$ are stopping times with $\sigma \leq \tau$, then we also have $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. Indeed, $\sigma \leq \tau$ implies that we have $\{ \tau \leq k \} = \{ \sigma \leq k, \tau \leq k \}$ for all $k$; so if $A \in \mathcal{F}_\sigma$ and $k$ is fixed, then

$$A \cap \{ \tau \leq k \} = (A \cap \{ \sigma \leq k \}) \cap \{ \tau \leq k \} \in \mathcal{F}_k$$

and hence also $A \in \mathcal{F}_\tau$. 

3
**Definition.** Let $Y = (Y_k)_{k \in \mathbb{N}_0}$ be an adapted stochastic process and $\tau$ a stopping time. The *value of $Y$ at time $\tau$* is then defined by

$$(Y_\tau)(\omega) := (Y_{\tau(\omega)})(\omega),$$

provided that $\tau$ is finite-valued (i.e. $\tau < \infty$ $P$-a.s.). The process $Y$ *stopped in $\tau$* is defined by $Y^\tau = (Y^\tau_k)_{k \in \mathbb{N}_0}$ with

$$Y^\tau_k(\omega) := Y_{k \wedge \tau}(\omega) = \begin{cases} Y_k(\omega) & \text{for } k \leq \tau(\omega) \\ Y_\tau(\omega) & \text{for } k > \tau(\omega) \end{cases}$$

and for $k \in \mathbb{N}_0$.

Note that $Y_\tau$ is a random variable, whereas $Y^\tau$ is a stochastic process and again adapted to $\mathcal{F}$. The measurability for both of these statements follows from the next result.

**Lemma C.2.** Suppose that $Y$ is an $\mathcal{F}$-adapted (real-valued) process and $\tau$ a finite-valued stopping time (with respect to $\mathcal{F}$). Then the mapping $Y_\tau : \Omega \to \mathbb{R}$ is $\mathcal{F}_\tau$-measurable.

**Proof.** We need to show that for every $c \in \mathbb{R}$, the set $\{Y_\tau \leq c\}$ is in $\mathcal{F}_\tau$. But for any fixed $k$, we have

$$\{Y_\tau \leq c\} \cap \{\tau \leq k\} = \bigcup_{j=0}^{k} \{Y_\tau \leq c, \tau = j\} = \bigcup_{j=0}^{k} \{Y_j \leq c, \tau = j\},$$

and for every $j \leq k$, we have both $\{Y_j \leq c\} \in \mathcal{F}_j \subseteq \mathcal{F}_k$ because $Y$ is adapted so that $Y_j$ is $\mathcal{F}_j$-measurable, and $\{\tau = j\} \in \mathcal{F}_j \subseteq \mathcal{F}_k$ because $\tau$ is a stopping time. Hence we obtain $\{Y_\tau \leq c\} \cap \{\tau \leq k\} \in \mathcal{F}_k$ for all $k$, and so $\{Y_\tau \leq c\} \in \mathcal{F}_\tau$. \textit{q.e.d.}

**Proposition C.3.** If $M$ is a martingale and $\tau$ a stopping time, then the stopped process $M^\tau$ is again a martingale. The same result is true for sub- and supermartingales.
Proof. Since we can argue for each coordinate, we may assume without loss of generality that $M$ is real-valued. Define $H_k := I_{\{k \leq \tau\}}$ for $k \in \mathbb{N}$. Then $H$ is bounded and predictable because $\{\tau \geq k\} = \{\tau \leq k - 1\}^c \in \mathcal{F}_{k-1}$ since $\tau$ is a stopping time. By Proposition C.1, the process

$$
\left( \int H \, dM \right)_k = \sum_{j=1}^k I_{\{j \leq \tau\}} \Delta M_j = \sum_{j=1}^{k \wedge \tau} \Delta M_j = M_{k \wedge \tau} - M_0 = M_k^\tau - M_0, \quad k \in \mathbb{N},$
$$
is therefore a martingale, and so is then $M^\tau$. The same argument also works for sub- and supermartingales because $H$ is nonnegative. \hfill \textbf{q.e.d.}

Definition. A stochastic process $M = (M_k)_{k \in \mathbb{N}}$ is called a \textit{local martingale (with respect to $P$ and $\mathcal{F}$)} if $M$ is $\mathcal{F}$-adapted and there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times with $\tau_n \not\nearrow \infty$ $P$-a.s. and such that for each $n$, the stopped process $M^{\tau_n}I_{\{\tau_n > 0\}}$ is a martingale. The sequence $(\tau_n)_{n \in \mathbb{N}}$ is then called a \textit{localising sequence}.

Remarks. 1) The indicator function $I_{\{\tau_n > 0\}}$ appears because one wants to avoid imposing any integrability conditions on $M_0$. If $M_0 = 0$ or if $M_0$ is nonrandom, one can equivalently ask that $M^{\tau_n}$ should be a martingale. This applies in particular in the frequently encountered case when $\mathcal{F}_0$ is trivial.

2) If $M$ is indexed by $k = 0, 1, \ldots, T$ with some $T \in \mathbb{N}$, the requirement for the localising sequence is that $\tau_n \not\nearrow T$ $P$-a.s. Because the time index is discrete, this is equivalent to the requirement that the sequence is increasing and $P[\tau_n < T] \to 0$ as $n \to \infty$.

3) Clearly every martingale is a local martingale; it is enough to take $\tau_n \equiv \infty$ (or $\tau_n \equiv T$ in the case of a finite time horizon $T$).

The notion of a local martingale allows us to extend Proposition C.1 from bounded to arbitrary predictable processes as integrands, at the mere cost of a localisation. Importantly, this result does not generalise to continuous time.
Proposition C.4. Suppose $M$ is an $\mathbb{R}^m$-valued local martingale and $H = (H_k)_{k \in \mathbb{N}}$ is an $\mathbb{R}^m$-valued predictable process. Then the stochastic integral $H \cdot M = \int H \, dM$ is again a local martingale.

Proof. Let $(\tau_n)$ be a localising sequence for $M$ and set

$$\sigma_n := \inf \{ j \in \mathbb{N}_0 \mid |H_{j+1}| > n \}.$$

Then $\{\sigma_n > k\} = \{|H_1| \leq n, \ldots, |H_{k+1}| \leq n\}$ is in $\mathcal{F}_k$ because $H$ is predictable, and so $\sigma_n$ is a stopping time. Moreover, $\sigma_n \nearrow \infty$ $P$-a.s. because $H$ is a finite-valued process. Therefore $\varrho_n := \sigma_n \wedge \tau_n$, $n \in \mathbb{N}$, is a sequence of stopping times with $\varrho_n \nearrow \infty$ and

$$\left( \int H \, dM \right)_k^{\varrho_n} = \sum_{j=1}^{k \wedge \varrho_n} H_j \cdot \Delta M_j$$

$$= \sum_{j=1}^{k} I_{\{j \leq \varrho_n\}} H_j \cdot \Delta M_j^{\varrho_n} I_{\{\varrho_n > 0\}}$$

$$= \left( \int (H \cdot I_{\{\cdot \leq \varrho_n\}}) \, d(M^{\varrho_n} I_{\{\varrho_n > 0\}}) \right), \quad k \in \mathbb{N}_0,$$

is a martingale by Proposition C.1, because $M^{\varrho_n} I_{\{\varrho_n > 0\}} = (M^{\tau_n} I_{\{\tau_n > 0\}})^{\sigma_n}$ is a martingale by Proposition C.3 and $H \cdot I_{\{\cdot \leq \varrho_n\}}$ is predictable and bounded by construction. This gives the result since $\int H \, dM$ is null at 0. \hspace{1cm} \textbf{q.e.d.}

The next result is very useful in many applications in mathematical finance in discrete time. We point out that it does not have any analogue in continuous time.

Theorem C.5. Suppose $L = (L_k)_{k \in \mathbb{N}_0}$ is a real-valued local martingale. If $E[|L_0|] < \infty$ and $E[L_T] < \infty$ for some $T \in \mathbb{N}$, then the stopped process $L^T = (L_k)_{k=0,1,\ldots,T}$ is a (true) martingale.

Proof. Let $(\tau_n)$ be a localising sequence for $L$. Then $I_{\{\tau_n > k-1\}} \nearrow 1$ $P$-a.s. as $n \to \infty$, for
every $k \in \mathbb{N}$.

1) We first show inductively that $E[L_k^-] < \infty$ for all $k = 1, \ldots, T - 1$. (For $k = 0$ and $k = T$, this holds by assumption.) Indeed, because $\{\tau_n > k - 1\} \in \mathcal{F}_{k-1}$ and $L^{\tau_n}I_{\{\tau_n > 0\}}$ is a martingale, the inequality $x^- \geq -x$ yields

$$E[L_k^- | \mathcal{F}_{k-1}]I_{\{\tau_n > k-1\}} = E[(L_k^{\tau_n})^-I_{\{\tau_n > 0\}} | \mathcal{F}_{k-1}]I_{\{\tau_n > k-1\}}$$

$$\geq -L_{k-1}^{\tau_n}I_{\{\tau_n > 0\}}I_{\{\tau_n > k-1\}}$$

$$= -L_{k-1}I_{\{\tau_n > k-1\}} \quad P\text{-a.s.}$$

Letting $n \to \infty$, we obtain

$$E[L_k^- | \mathcal{F}_{k-1}] \geq \max(0, -L_{k-1}) = L_{k-1}^- \quad P\text{-a.s.}$$

and therefore $E[L_{k-1}^-] \leq E[L_k^-]$. This gives the assertion above because $E[L_T^-] < \infty$ by assumption.

2) We next show that $E[|L_k|] < \infty$ for all $k = 1, \ldots, T$ so that the stopped process $L^T$ is integrable. Indeed, using $\tau_n \not\to \infty$, Fatou’s lemma and the martingale property of $L^{\tau_n}I_{\{\tau_n > 0\}}$ gives

$$E[L_k^+] = E\left[\lim_{n \to \infty} L_{k \wedge \tau_n}^+ I_{\{\tau_n > 0\}}\right]$$

$$\leq \liminf_{n \to \infty} E[L_{k \wedge \tau_n}^+ I_{\{\tau_n > 0\}}]$$

$$= \liminf_{n \to \infty} E[L_{k}^{\tau_n} I_{\{\tau_n > 0\}} + L_{k \wedge \tau_n}^- I_{\{\tau_n > 0\}}]$$

$$= E[L_0 I_{\{\tau_n > 0\}}] + \liminf_{n \to \infty} E[L_{k \wedge \tau_n}^- I_{\{\tau_n > 0\}}].$$

By Step 1), the sum $\sum_{j=0}^{T} L_j^-$ is an integrable majorant for every $L_{k \wedge \tau_n}$ so that we obtain directly

$$E[L_k^+] \leq E[|L_0|] + \sum_{j=0}^{T} E[L_j^-] < \infty$$

by the assumption and Step 1).
3) To show the martingale property of $L$, we note that for all $k = 0, 1, \ldots, T$ and $n \in \mathbb{N}$,

$$|L^n_k|I_{\{\tau_n > 0\}} \leq \max_{j=0, \ldots, T} |L_j| \leq \sum_{j=0}^{T} |L_j| \in L^1(P)$$

by Step 2). Moreover, $L^n \Gamma_{\{\tau_n > 0\}}$ is a martingale so that we obtain by dominated convergence for $k \geq 1$ that

$$E[L_k | \mathcal{F}_{k-1}] = E \left[ \lim_{n \to \infty} L_{k \wedge \tau_n} I_{\{\tau_n > 0\}} \middle| \mathcal{F}_{k-1} \right]$$

$$= \lim_{n \to \infty} E[L_{k \wedge \tau_n} I_{\{\tau_n > 0\}} \middle| \mathcal{F}_{k-1}]$$

$$= \lim_{n \to \infty} L_{(k-1) \wedge \tau_n} I_{\{\tau_n > 0\}}$$

$$= L_{k-1} \quad P\text{-a.s.}$$

This completes the proof. \textit{q.e.d.}

**Corollary C.6.**

1) Suppose $L$ is a real-valued local martingale with $E[|L_0|] < \infty$ and $L \geq -a$ for some $a \geq 0$, in the sense that $L_k \geq -a$ $P$-a.s. for all $k \in \mathbb{N}_0$. Then $L = (L_k)_{k \in \mathbb{N}_0}$ is a (true) martingale.

2) Suppose $M$ is an $\mathbb{R}^m$-valued local martingale. For any $\mathbb{R}^m$-valued predictable process $H = (H_k)_{k \in \mathbb{N}}$ with $\int H \, dM \geq -a$ for some constant $a \geq 0$, the stochastic integral process $H \cdot M = \int H \, dM$ is a (true) martingale.

**Proof.**

1) This follows directly from Theorem C.5 because $L^n_T \in L^1(P)$ for every $T \in \mathbb{N}$.

2) We know from Proposition C.4 that $L := H \cdot M$ is a real-valued local martingale. So we can apply part 1) to get the result. \textit{q.e.d.}

**Remark.** Imposing the (boundedness or integrability) condition on $L^-$ is natural in the context of mathematical finance, as we shall see later. However, from a purely mathematical perspective, we could equally well impose the analogous condition on $L^+$ and obtain the same conclusion by considering $-L$ instead of $L$. 
The next result is a very convenient characterisation of martingales in finite discrete time.

**Lemma C.7.** Let \( Y = (Y_k)_{k=0,1,...,T} \) be an \( \mathbb{R}^m \)-valued adapted integrable stochastic process. Then the following are equivalent:

1) \( Y \) is a martingale.

2) \( E[(\int H \, dY)_T] = 0 \) for all \( \mathbb{R}^m \)-valued bounded predictable processes \( H = (H_k)_{k=1,...,T} \).

3) \( E[Y_\tau] = E[Y_0] \) for all stopping times \( \tau \) with values in \( \{0,1,\ldots,T\} \).

**Proof.** “1) \( \Rightarrow \) 2)”: For every \( H \) as in 2), \( L := \int H \, dY \) is a martingale by Proposition C.1, and so \( E[L_T] = E[L_0] \).

“2) \( \Rightarrow \) 3)”): From the proof of Proposition C.3, we can see that for each coordinate, \( (Y^i)_\tau - Y^i_0 = \int H \, dY \) for some bounded predictable \( H \). So the assertion follows because \( (Y^i)_T = Y^i_\tau \).

“3) \( \Rightarrow \) 1)”): By arguing separately for each coordinate, we can assume without loss of generality that \( n = 1 \). We show that \( E[Y_\tau \mid \mathcal{F}_k] = Y_k \) for \( k = 0,1,\ldots,T \) by choosing a suitable stopping time \( \tau \). Fix \( k \) and \( A \in \mathcal{F}_k \) and define \( \tau := kI_A + TI_{A^c} \). Then \( \tau \) is a stopping time because

\[
\{\tau \leq \ell\} = (\{k \leq \ell\} \cap A) \cup (\{T \leq \ell\} \cap A^c) = \begin{cases} \Omega & \text{for } \ell = T \\ A \in \mathcal{F}_k \subseteq \mathcal{F}_\ell & \text{for } k \leq \ell < T \\ \emptyset & \text{for } k > \ell \end{cases}
\]

is always in \( \mathcal{F}_\ell \). Because \( T \) is also a stopping time, we obtain

\[
E[Y_\tau] = E[Y_0] = E[Y_T]
\]

and therefore

\[
E[Y_k I_A] = E[Y_T I_A]
\]

by the definition of \( \tau \). Since this holds for any \( A \in \mathcal{F}_k \) and \( Y_k \) is \( \mathcal{F}_k \)-measurable, we obtain

\[
Y_k = E[Y_T \mid \mathcal{F}_k] \quad \text{P-a.s.}
\]
and this proves the result. \[q.e.d.\]

If \( M \) is a martingale, we have

\[ E[M_{k} \mid \mathcal{F}_k] = M_k \quad \text{P-a.s. for } k \geq k. \]

We now want to show that this still holds if we replace the deterministic times \( k \leq \ell \) by bounded stopping times \( \sigma \leq \tau \).

**Theorem C.8. (Stopping theorem)** Suppose \( M = (M_k)_{k \in \mathbb{N}_0} \) is a martingale and \( \sigma, \tau \) are stopping times with \( \sigma \leq \tau \leq T \) P-a.s. for some \( T \in \mathbb{N} \). Then

\[ E[M_\tau \mid \mathcal{F}_\sigma] = M_\sigma \quad \text{P-a.s.}, \]

i.e., the martingale property also holds at (bounded) stopping times.

**Proof.** By looking at the stopped process \( M_T \) and using that \( \sigma, \tau \) are bounded by \( T \), we can assume without loss of generality that \( M \) is only indexed by \( k = 0, 1, \ldots, T \). Moreover, both \( M_\tau = \sum_{k=0}^{T} M_k I_{\{\tau = k\}} \) and \( M_\sigma \) are integrable so that the conditional expectation is well defined. Because of \( \sigma \leq \tau \), we have \( \mathcal{F}_\sigma \subseteq \mathcal{F}_\tau \), and so it is enough to prove the case where \( \tau = T \); indeed, if we consider the pairs \((\sigma, T)\) and \((\tau, T)\), we obtain from the projectivity of conditional expectations that

\[ M_\sigma = E[M_T \mid \mathcal{F}_\sigma] = E[E[M_T \mid \mathcal{F}_\tau] \mid \mathcal{F}_\sigma] = E[M_\tau \mid \mathcal{F}_\sigma] \quad \text{P-a.s.} \]

So assume that \( \tau = T \). Because \( M_\sigma \) is \( \mathcal{F}_\sigma \)-measurable by Lemma C.2, we only need to prove that \( E[M_T I_A] = E[M_\sigma I_A] \) for any \( A \in \mathcal{F}_\sigma \). But if \( A \in \mathcal{F}_\sigma \), then \( A \cap \{\sigma = k\} \in \mathcal{F}_k \) for all \( k \) and therefore, using the martingale property \( E[M_T \mid \mathcal{F}_k] = M_k \) and that \( M_k = M_\sigma \) on \( \{\sigma = k\} \), we obtain

\[ E[M_T I_A] = \sum_{k=0}^{T} E[M_T I_{A \cap \{\sigma = k\}}] = \sum_{k=0}^{T} E[M_k I_{A \cap \{\sigma = k\}}] = E[M_\sigma I_A]. \]
This completes the proof. \textbf{q.e.d.}

\textbf{Remark.} We have imposed the assumption that $\sigma, \tau$ are bounded because we shall mostly work in a setting of finite discrete time. There are other versions of the stopping theorem which obtain the same conclusion under different conditions on $\tau$ and/or $M$. Without any conditions except the martingale and stopping time properties, however, the result is not true.