Introduction to Mathematical Finance

Exercise sheet 1

Please hand in your solutions until Friday, 28/02/2020, 13:00 into Bálint Gersey's box next to HG G 53.2.

Exercise 1.1 Let $\mathcal{C} := \mathbb{R} \times \mathbb{R}^K$ be the consumption space with the payoff matrix $\mathcal{D}$ and let e^i, π be an endowment, and a price vector, respectively. Recall the budget set

$$B(e^i, \pi) := \{c \in \mathcal{C} : \exists \vartheta \in \mathbb{R}^N \text{ with } c_0 \leq e^i_0 - \vartheta \cdot \pi \text{ and } c_T \leq e^i_T + \mathcal{D}\vartheta\}.$$

(a) Show $c \in B(e^i, \pi) \iff c - e^i \in B(0, \pi) \Longleftarrow c - e^i$ is attainable with 0 initial wealth.

(b) Show by an example that the converse of the second implication is not true in general.

Solution 1.1

(a) By definition, $c \in B(e^i, \pi)$ iff there exists $\vartheta \in \mathbb{R}^N$ with $c_0 \leq e^i_0 - \vartheta \cdot \pi$ and $c_T \leq e^i_T + \mathcal{D}\vartheta$. That is, $c_0 - e^i_0 \leq -\vartheta \cdot \pi$ and $c_T - e^i_T \leq \mathcal{D}\vartheta$, which means $c - e \in B(0, \pi)$.

Now if $c - e^i$ is attainable with 0 initial wealth, then there exists $\hat{\vartheta} \in \mathbb{R}^N$ such that $c_0 - e^i_0 = -\pi \cdot \hat{\vartheta}$ and $c_T - e^i_T = \mathcal{D}\hat{\vartheta}$ which shows $c - e^i \in B(0, \pi)$.

(b) The idea is simply to find a nonattainable consumption which still lies in the budget set. To do this, we consider a matrix without full rank. Let

$$\pi := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathcal{D} := \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Clearly $\mathcal{D}(\mathbb{R}^2) = \{(a, 2a)^{\mathrm{tr}} : a \in \mathbb{R}\}$. Take for instance $\vartheta = (1, 0)^{\mathrm{tr}}$, $c_T = e^i_T + (1, 1.5)^{\mathrm{tr}}$, and $c_0 = e^i_0 - 1$. Then

$$c_0 - e^i_0 \leq -(1, 0) \cdot (1, 1) = -1,$$

$$c_T - e^i_T \leq \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, $c - e^i \in B(0, \pi)$. But clearly $(1, 1.5)^{\mathrm{tr}} \notin \mathcal{D}(\mathbb{R}^2)$, which shows $c - e^i$ cannot be attainable with 0 initial wealth.

Exercise 1.2

(a) Construct a market with arbitrage of the first kind but with no arbitrage of the second kind.

(b) Construct a market with arbitrage of the second kind but with no arbitrage of the first kind.

(c) Prove Proposition I.3.1. That is suppose there exists an asset $\mathcal{D}^l$ with $\mathcal{D}^l \geq 0$ and $\mathcal{D}^l \not\equiv 0$. Show that under this assumption, the market is arbitrage-free iff there is no arbitrage of first kind.

Solution 1.2

(a) Consider a market consisting of a single asset with $\pi = 0$, $\mathcal{D} = (1,2)^{\mathrm{tr}}$. Set $\vartheta = 1$. Clearly, $\mathcal{D}\vartheta = (1,2)^{\mathrm{tr}} \geq 0$ and $\mathcal{D}\vartheta(\{\omega_i\}) > 0$ for both $i = 1, 2$. Thus ϑ is an arbitrage opportunity of the first kind. However, since $\pi = 0$, there exists no arbitrage of the second kind.

(b) Consider the situation where $\pi = 1$ and $\mathcal{D} = (0,0)$. Then $\vartheta < 0$ would be an arbitrage of the second kind. But since $\mathcal{D}$ vanishes, we have for any $\tilde{\vartheta} \in \mathbb{R}$ that $\mathcal{D}\tilde{\vartheta} = (0,0)^{\mathrm{tr}}$. So there exists no arbitrage of the first kind.

(c) Suppose first that there is an asset $D^\ell \geq 0$ and $D^\ell \not\equiv 0$ and $\pi^\ell > 0$. Let ϑ be an arbitrage opportunity of the second kind. Set $\alpha = -\vartheta \cdot \pi / \pi^\ell > 0$. We consider a new strategy $\hat{\vartheta} = \vartheta + \alpha e_\ell$ where e_ℓ is the vector with 1 in its ℓth component and 0 elsewhere. Then $\hat{\vartheta} \cdot \pi = \vartheta \cdot \pi + \alpha \cdot \pi^\ell = 0$ and $\mathcal{D}\hat{\vartheta} = \mathcal{D}\vartheta + \alpha\mathcal{D}^\ell \geq 0$. Since $\mathcal{D}\vartheta \geq 0$ and $\alpha\mathcal{D}^\ell \geq 0$ with $\alpha\mathcal{D}^\ell \not\equiv 0$, we have $\mathcal{D}\hat{\vartheta} \geq 0$ and $\mathcal{D}\hat{\vartheta} \not\equiv 0$. Hence, $\hat{\vartheta}$ is an arbitrage opportunity of the first kind. The other implication is true in general.

Exercise 1.3 Let $\succeq$ be a preference order on $\mathcal{C}$ satisfying axioms (P1)-(P5). A function $\mathcal{U} : \mathcal{C} \to \mathbb{R}$ is called a *utility functional representing* $\succeq$ or *a numerical representation of* $\succeq$ if

$$c' \succeq c \quad \Longleftrightarrow \quad \mathcal{U}(c') \geq \mathcal{U}(c).$$

(a) Show that all $\mathcal{U}$ representing $\succeq$ must be *quasiconcave*, i.e., for all $c, c' \in \mathcal{C}$ and $\lambda \in [0,1]$,

$$\mathcal{U}\left(\lambda c + (1-\lambda)c'\right) \geq \min\{\mathcal{U}(c), \mathcal{U}(c')\}.$$

(b) Which axioms are needed for this result?

(c) Show by a counterexample that a preference order can be represented by a utility functional which is not concave.

Solution 1.3

(a) Let c' and c be arbitrary elements of $\mathcal{C}$. Without loss of generality, assume that $c' \succeq c$. Then, by convexity, $\lambda c' + (1-\lambda)c \succeq c$, and hence

$$\mathcal{U}\left(\lambda c' + (1-\lambda)c\right) \geq \mathcal{U}(c) = \min\{\mathcal{U}(c), \mathcal{U}(c')\}.$$

(b) In the solution above, we implicitly used completeness to assume $c' \succsim c$, and we used convexity directly.

(c) Define $\succeq$ by

$$c' \succeq c \quad :\Longleftrightarrow \quad c' \cdot \mathbf{1} \geq c \cdot \mathbf{1}.$$

It is easy to check that this satisfies the axioms (P1)–(P4). The natural utility functional is then given by

$$\mathcal{U}(c) = c \cdot \mathbf{1}.$$

However, since $\exp(\cdot)$ is increasing, it will preserve the order. Hence, $\exp(\mathcal{U}(\cdot))$ is also a utility functional, but not concave. More generally, exp can be replaced by any strictly increasing function on $\mathbb{R}$.

Exercise 1.4 *This question is **optional**.*

(a) Show that any complete and transitive relation $\succeq$ induces an asymmetric and negative transitive order $\succ$ via

$$y \succ x \iff x \not\succeq y$$

Conversely, show that any asymmetric and negative transitive binary relation $\succ$ induces a complete and transitive binary relation $\succeq$.

In this question, we refer to an asymmetric and negative transitive relation $\succ$ as *preference order* and to the corresponding complete and transitive binary relation $\succeq$ as *weak preference order*. Moreover, we denote by $\mathcal{C}$ the set of consumption processes.

(b) Does every function $U : \mathcal{C} \to \mathbb{R}$ represent some preference order, i.e. an asymmetric and negative transitive relation?

Let $\succ$ be a preference relation on $\mathcal{C}$. A subset $\mathcal{Z}$ of $\mathcal{C}$ is called *order dense* if for any pair $x, y \in \mathcal{C}$ such that $x \succ y$ there exists some $z \in \mathcal{Z}$ with $x \succeq z \succeq y$.

(c) Show that, for the existence of a numerical representation of a preference relation $\succ$, it is necessary and sufficient that $\mathcal{C}$ contains a countable, order dense subset $\mathcal{Z}$.

(d) Find a preference order that does not admit a numerical representation. Which axioms from (P1)-(P5) does your example not satisfy?
Hint: Try the lexicographical order

Solution 1.4
This exercise closely follows Chapter 2 of "Stochastic Finance – An Introduction in Discrete Time" by Hans Föllmer and Alexander Schied.

(a) Let $\succeq$ be a binary relation satisfying

1. *Completeness:* for all $x, y \in \mathcal{C}$ $\quad x \succeq y$ or $y \succeq x$
2. *Transitivity:* if $x \succeq y$ and $y \succeq z$ then $x \succeq z$

We want to show that the binary relation $\succ$ defined as $y \succ x \iff x \not\succeq y$ satisfies

1. *Assymetry:* if $x \succ y$ then $y \not\succ x$
2. *Negative transitivity:* if $x \succ y$ and $z \in \mathcal{C}$ then either $x \succ z$ or $z \succ y$ or both must hold

The proofs are trivial and only use the definitions. First, let $\succeq$ be a complete and transitive relation. We show that the corresponding $\succ$ is asymmetric and negative transitive.

- Suppose $x \succ y$. We want to show $y \not\succ x$, i.e $x \succeq y$. This is clear because by completeness of $\succeq$ we have $x \succeq y$ or $y \succeq x$, but $y \succeq x$ cannot be true since $x \succ y \iff y \not\succeq x$.
- Let $x \succ y$ and $z \in \mathcal{C}$. We need to show that either $x \succ z$ or $z \succ y$. By contradiction, suppose that $x \not\succ z$ and $z \not\succ y$, which by definition is equivalent to $z \succeq x$ and $y \succeq z$. By transitivity, we then have $y \succeq x$ which contradicts $x \succ y$.

Conversely let $\succ$ be an asymmetric and negative transitive binary relation. We show that the corresponding $\succeq$ is complete and transitive.

- By contradiction, suppose $y \not\succeq x$ and $x \not\succeq y$. By definition this is equivalent to $x \succ y$ and $y \succ x$ which contradicts the asymmetry of $\succ$.
- Let $x, y, z \in \mathcal{C}$ be such that $x \succeq y$ and $y \succeq z$. We want to show $x \succeq z$. By contradiction, suppose that $x \not\succeq z$, i.e. $z \succ x$. By negative transitivity, we must have either $z \succ y$ or $y \succ x$. But none of them is possible, as $x \succeq y$ and $y \succeq z$.

(b) Yes, every function $U : \mathcal{C} \to \mathbb{R}$ does represent an asymmetric and negative transitive binary relation. Indeed, given a function $U : \mathcal{C} \to \mathbb{R}$, consider the binary relation

$$x \succ_U y \iff U(x) > U(y)$$

or, equivalently,

$$x \succeq_U y \iff U(x) \geq U(y)$$

We need to show that $\succeq_U$ is complete and transitive.

- Clearly, for all $x, y \in \mathcal{C}$, we have either $U(x) \geq U(y)$ or $U(y) \geq U(x)$ and hence $x \succeq_U y$ or $y \succeq_U x$.
- Suppose $x \succeq_U y$ and $y \succeq_U z$, i.e. $U(x) \geq U(y)$ and $U(y) \geq U(z)$. By transitivity of $\geq$, we have $U(x) \geq U(z)$ and hence $x \succeq_U z$.

(c) Suppose first that we are given a countable order dense subset $\mathcal{Z}$ of $\mathcal{C}$. For $x \in \mathcal{C}$, set

$$\mathcal{Z}_\succ(x) := \{z \in \mathcal{Z} | z \succ x\} \quad \text{and} \quad \mathcal{Z}_\prec(x) := \{z \in \mathcal{Z} | x \succ z\}.$$

The relation $x \succeq y$ implies that $\mathcal{Z}_\succ(x) \subseteq \mathcal{Z}_\succ(y)$ and $\mathcal{Z}_\prec(x) \supseteq \mathcal{Z}_\prec(y)$. If the strict relation $x \succ y$ holds, then at least one of these inclusions is also strict. Indeed, using that $\mathcal{Z}$ is order dense in $\mathcal{C}$, we can pick $z \in \mathcal{Z}$ with $x \succeq z \succeq y$, so either $x \succ z \succeq y$ or $x \succeq z \succ y$. In the first case $z \in \mathcal{Z}_\prec(x) \setminus \mathcal{Z}_\prec(y)$, while $z \in \mathcal{Z}_\succ(y) \setminus \mathcal{Z}_\succ(x)$ in the second case.

To construct a numerical representation U of $\succ$, consider any strictly positive probability measure μ on $\mathcal{Z}$, and let

$$U(x) := \sum_{z \in \mathcal{Z}_{\prec}(x)} \mu(z) - \sum_{z \in \mathcal{Z}_{\succ}(x)} \mu(z)$$

The above arguments show that $U(x) > U(y)$ if and only if $x \succ y$ and hence U is a desired numerical representation.

For the proof of the converse assertion, take a numerical representation U and let $\mathcal{J}$ denote the countable set

$$\mathcal{J} := \{[a,b] | a,b \in \mathbb{Q}, a < b, U^{-1}([a,b]) \neq \emptyset\}$$

For every interval $I \in \mathcal{J}$, we can choose some $z_I \in \mathcal{C}$ with $U(z_I) \in I$ and thus define the countable set

$$A := \{z_I | I \in \mathcal{J}\}$$

At first glance it may seem that A is a good candidate for an order dense set. However, it may happen that there are $x, y \in \mathcal{C}$ such that $U(x) < U(y)$ and for which there is no $z \in \mathcal{C}$ with $U(x) < U(z) < U(y)$. In this case, an order dense set must contain at least one z with $U(z) = U(x)$ or $U(z) = U(y)$, a condition which cannot be guaranteed by A.

Let us define the set $\mathcal{D}$ of all pairs (x,y) which do not admit any $z \in A$ with $y \succ z \succ x$:

$$\mathcal{D} = \{(x,y) | x, y \in \mathcal{C} \setminus A,\ y \succ x \text{ and } \nexists z \in A \text{ with } y \succ z \succ x\}.$$

Note that $(x,y) \in \mathcal{D}$ implies that we cannot find $z \in \mathcal{C}$ with $y \succ z \succ x$. Indeed using the density of rational numbers, we could then find $a, b \in \mathbb{Q}$ such that

$$U(x) < a < U(z) < b < U(y),$$

so $I := [a,b]$ would belong to $\mathcal{J}$, and the corresponding z_I would be an element of A satisfying $y \succ z_I \succ z$, contradicting the assumption that $(x,y) \in \mathcal{D}$.

It follows that all intervals $(U(x), U(y))$ with $(x,y) \in \mathcal{D}$ are disjoint and nonempty. Hence, there can only be countably many of them. For each such interval J, we choose exactly one pair $(x^J, y^J) \in \mathcal{D}$ such that $U(x^J)$ and $U(y^J)$ are the endpoints of the interval J, and we denote B the countable set containing all x^J and y^J.

It remains to show that $\mathcal{Z} := A \cup B$ is an order dense subset of $\mathcal{C}$. Let $x, y \in \mathcal{C} \setminus \mathcal{Z}$ with $y \succ x$. Then, exactly one of the following hold. Either there is some $z \in A$ such that $y \succ z \succ x$, or $(x,y) \in \mathcal{D}$. In the latter case, there exists some $z \in B$ with $U(y) = U(z) > U(x)$ and consequently $y \succeq z \succ x$. Moreover the set $\mathcal{Z}$ is by construction countable which finishes the proof.

(d) Let $\succ$ be the usual lexicographical order on $\mathcal{C} := [0,1] \times [0,1]$, i.e. $(x_1, x_2) \succ (y_1, y_2)$ if and only if either $x_1 > y_1$ or $x_1 = y_1$ and simultaneously $x_2 > y_2$. It is easy to verify (left as exercise) that $\succ$ is asymmetric and negative transitive, and hence a preference order. We show that $\succ$ does not admit a numerical representation. To this end, let $\mathcal{Z}$ be any order dense subset of $\mathcal{C}$. Then for $x \in [0,1]$ there must exist some $(z_1, z_2) \in \mathcal{Z}$ such that

$$(x, 1) \succeq (z_1, z_2) \succeq (x, 0)$$

It follows that $z_1 = x$ and hence $\mathcal{Z}$ is uncountable. The result of the previous question therefore implies that the lexicographical order cannot have a numerical representation.

Recall that a weak preference order $\succeq$ is called continuous if the sets

$$\mathcal{B}_{\succeq}(x) := \{y \in \mathcal{C} | y \succeq x\} \quad \text{and} \quad \mathcal{B}_{\preceq}(x) := \{y \in \mathcal{C} | x \succeq y\}$$

are closed for all $x \in \mathcal{C}$. Alternatively we can define continuity in terms of the corresponding preference order $\succ$. We say that $\succ$ is continuous if for all $x \in \mathcal{C}$ the sets

$$\mathcal{B}_{\succ}(x) := \{y \in \mathcal{C} | y \succ x\} \quad \text{and} \quad \mathcal{B}_{\prec}(x) := \{y \in \mathcal{C} | x \succ y\}$$

are open. We next show that the lexicographical order is not continuous. Indeed for any given $(x_1, x_2) \in [0,1] \times [0,1]$, the set

$$\{(y_1, y_2) | (y_1, y_2) \succ (x_1, x_2)\} = (x_1, 1] \times [0,1] \cup \{x_1\} \times (x_2, 1]$$

is not open.