Introduction to Mathematical Finance

Exercise sheet 10

Exercise 10.1 (Mean-variance hedging). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Suppose that the discounted price process X satisfies $E[(\triangle X_k)^2 | \mathcal{F}_{k-1}] < \infty$ P-a.s. for all k. Define

$$\mathcal{A} := \Big\{ \text{all predictable processes } \vartheta = (\vartheta_k)_{k=1,\dots,T} : (\vartheta \bullet X)_k \in L^2 \text{ for } k = 1, \dots, T \Big\}.$$

Let $c \in \mathbb{R}$ and $H \in L^2(\mathcal{F}_T)$. Mean-variance hedging (MVH) is the problem of approximating, with minimal mean squared error, a given payoff by the final value of a self-financing trading strategy in a financial market. We thus consider the problem

$$\min_{\vartheta \in \mathcal{A}} E[(H - c - (\vartheta \bullet X)_T)^2] \tag{1}$$

The goal of this exercise is to construct a candidate for the optimal strategy using the MOP. For $\vartheta \in \mathcal{A}$, we set

$$\begin{aligned} \mathcal{A}_k(\vartheta) &:= \{\vartheta' \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \leq k\}, \\ \mathcal{A}_k &:= \mathcal{A}_k(0) = \{\vartheta' \in \mathcal{A} : \vartheta'_j = 0 \text{ for } j \leq k\}. \end{aligned}$$

For $v_k \in L^2(\mathcal{F}_k)$, we define

$$\begin{aligned} \Gamma_k(v_k, \vartheta') &:= E\bigg[\Big(H - v_k - \sum_{j=k+1}^{T} \vartheta'_j \triangle X_j \Big)^2 \bigg| \mathcal{F}_k \bigg], \\ V_k(v_k) &:= \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_k} \Gamma_k(v_k, \vartheta'). \end{aligned}$$

(a) Show that for each k and each $v_k \in L^2(\mathcal{F}_k)$, the collection of random variables

$$\Lambda_k(v_k) := \Big\{ \Gamma_k(v_k, \vartheta') : \vartheta' \in \mathcal{A}_k(0) \Big\}$$

is closed under taking minima.

(b) Show that for fixed $\vartheta \in \mathcal{A}$, $x \in \mathbb{R}$, the process $(V_k(x + (\vartheta \bullet X)_k))_{k=0,\dots,T}$ is a submartingale. *Hint: Use Corollary 2 from Appendix E*

(c) Show that $\vartheta^* \in \mathcal{A}$ is optimal if and only if the process $(V_k(c + (\vartheta^* \bullet X)_k))_{k=0,\dots,T}$ is a martingale.

(d) Show that (V_k) satisfies the recursion

$$V_{k-1}(x) = \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_{k-1}} E[V_k(x + \vartheta'_k \triangle X_k) | \mathcal{F}_{k-1}]$$

with $V_T(x) = (H - x)^2$.

Solution 10.1

(a) Let $\vartheta^1, \vartheta^2 \in \mathcal{A}_k(0)$. Define

$$\vartheta^3 := \vartheta^1 \mathbb{1}_A + \vartheta^2 \mathbb{1}_{A^c},$$

where $A := \{\Gamma_k(v_k, \vartheta^1) \leq \Gamma_k(v_k, \vartheta^2)\}$. We have to show that $\vartheta^3 \in \mathcal{A}_k(0)$ and $\Gamma_k(v_k, \vartheta^3) = \min\left\{\Gamma_k(v_k, \vartheta^1), \Gamma_k(v_k, \vartheta^2)\right\}$. Since $\vartheta^1 \in \mathcal{A}_k(0)$ and $\vartheta^2 \in \mathcal{A}_k(0)$ we clearly have $\vartheta^3_j = 0$ for $j \leq k$. Moreover, using that $(\vartheta^i \bullet X)_k \in L^2$ for $i \in \{1, 2\}$ and $(\vartheta^3 \bullet X)_k = \mathbb{1}_A (\vartheta^1 \bullet X)_k + \mathbb{1}_{A^c} (\vartheta^2 \bullet X)_k$, we have $(\vartheta^3 \bullet X)_k \in L^2(\mathcal{F}_k)$ for each k. This gives $\vartheta^3 \in \mathcal{A}_k(0)$. Further note that

$$\begin{aligned} H - v_k - \sum_{j=k+1}^{T} \vartheta^3_j \triangle X_j = \mathbb{1}_A \Big(H - v_k - \sum_{j=k+1}^{T} \vartheta^1_j \triangle X_j \Big) \\ + \mathbb{1}_{A^c} \Big(H - v_k - \sum_{j=k+1}^{T} \vartheta^2_j \triangle X_j \Big) \end{aligned}$$

is also in $L^2(\mathcal{F}_T)$ for each k and hence $\Gamma_k(v_k, \vartheta^3)$ is well-defined . Finally, since $A \in \mathcal{F}_k$, we obtain

$$\begin{aligned} \Gamma_k(v_k, \vartheta^3) &= E\Big[\Big(H - v_k - \sum_{j=k+1}^{T} \vartheta^3_j \triangle X_j \Big)^2 \Big| \mathcal{F}_k \Big] \\ &= \mathbb{1}_A \Gamma_k(v_k, \vartheta^1) + \mathbb{1}_{A^c} \Gamma_k(v_k, \vartheta^2) \\ &= \min\left\{ \Gamma_k(v_k, \vartheta^1), \Gamma_k(v_k, \vartheta^2) \right\}. \end{aligned}$$

(b) Fix $k \leq \ell$. We apply part (a) with $v_k = x + (\vartheta \bullet X)_k \in L^2(\mathcal{F}_k)$. So Corollary E.2 yields

$$\begin{aligned} V_\ell(x + (\vartheta \bullet X)_\ell) &= \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_\ell(0)} \Gamma_\ell(x + (\vartheta \bullet X)_\ell, \vartheta') \\ &= \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_\ell(0)} E\Big[\Big(H - x - \sum_{j=1}^{\ell} \vartheta_j \triangle X_j - \sum_{j=\ell+1}^{T} \vartheta'_j \triangle X_j \Big)^2 \Big| \mathcal{F}_\ell \Big] \\ &=\downarrow \lim_{n \to \infty} E\Big[\Big(H - x - \sum_{j=1}^{\ell} \vartheta_j \triangle X_j - \sum_{j=\ell+1}^{T} \vartheta^n_j \triangle X_j \Big)^2 \Big| \mathcal{F}_\ell \Big] \end{aligned}$$

for a sequence $(\vartheta^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_\ell(0) \subseteq \mathcal{A}_k(0)$. Note that $\Gamma_\ell(x+\vartheta \bullet X_\ell, \vartheta^n)$ is in L^1 due to the definitions of $\vartheta, (\vartheta^n)_{n\in\mathbb{N}}$. Then using monotone convergence, the tower property and $(\vartheta^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_\ell(0) \subseteq \mathcal{A}_k(0)$, we have

$$\begin{aligned}
E[V_\ell(x+(\vartheta\bullet X)_\ell)|\mathcal{F}_k] &= E\Big[\lim_{n\to\infty} E\Big[\Big(H - x - \sum_{j=1}^{\ell}\vartheta_j\triangle X_j - \sum_{j=\ell+1}^{T}\vartheta_j^n\triangle X_j\Big)^2\Big|\mathcal{F}_\ell\Big]\Big|\mathcal{F}_k\Big] \\
&= \lim_{n\to\infty} E\Big[E\Big[\Big(H - x - \sum_{j=1}^{\ell}\vartheta_j\triangle X_j - \sum_{j=\ell+1}^{T}\vartheta_j^n\triangle X_j\Big)^2\Big|\mathcal{F}_\ell\Big]\Big|\mathcal{F}_k\Big] \\
&= \lim_{n\to\infty} E\Big[\Big(H - x - \sum_{j=1}^{k}\vartheta_j\triangle X_j - \sum_{j=k+1}^{T}\vartheta_j^n\triangle X_j\Big)^2\Big|\mathcal{F}_k\Big] \\
&\geq \operatorname*{ess\,inf}_{\vartheta'\in\mathcal{A}_k(0)} E\Big[\Big(H - x - \sum_{j=1}^{k}\vartheta_j\triangle X_j - \sum_{j=k+1}^{T}\vartheta_j'\triangle X_j\Big)^2\Big|\mathcal{F}_k\Big] \\
&= V_k(x+(\vartheta\bullet X)_k),
\end{aligned}$$

and so we have the submartingale property. The integrability then follows from

$$V_T(x+(\vartheta\bullet X)_T) = (H - x - (\vartheta\bullet X)_T)^2 \in L^1.$$

Remark: note that V_k is a non-negative random variable by definition of Γ_k

(c) "$\Rightarrow$" Let $\vartheta^* \in \mathcal{A}$ be optimal. We already know that $(V_k(x+(\vartheta^*\bullet X)_k))_{k=0,\dots,T}$ is a submartingale. To show that it is a martingale, we thus only need to show that

$$E[V_T(c+(\vartheta^*\bullet X)_T)] = E[V_0(c)].$$

By the optimality of ϑ^*, we have as in the lecture

$$\begin{aligned}
E[V_0(c)] &= E\Big[\operatorname*{ess\,inf}_{\vartheta\in\mathcal{A}_0} E[(H-c-(\vartheta\bullet X)_T)^2|\mathcal{F}_0]\Big] \\
&= \inf_{\vartheta\in\mathcal{A}} E[(H-c-(\vartheta\bullet X)_T)^2] \\
&= E[(H-c-(\vartheta^*\bullet X)_T)^2] = E[V_T(c+(\vartheta^*\bullet X)_T)].
\end{aligned}$$

This gives the desired equality.

"$\Leftarrow$" Suppose that $(V_k(x+(\vartheta^*\bullet X)_k))_{k=0,\dots,T}$ is a martingale. Then using $V_T(c+(\vartheta^*\bullet X)_T) = (H-c-(\vartheta^*\bullet X)_T)^2$ gives

$$E[V_0(c)] = E[V_T(c+(\vartheta^*\bullet X)_T)] = E[(H-c-(\vartheta^*\bullet X)_T)^2]$$

Moreover, the same argument as above shows that

$$E[V_0(C)] = \inf_{\vartheta\in\mathcal{A}} E[(H-c-(\vartheta\bullet X)_T)^2],$$

which implies that ϑ^* is optimal.

(d) By part (b), we have for every fixed $\vartheta' \in \mathcal{A}_{k-1}$ that the process $V_\cdot(x + (\vartheta' \bullet X)_\cdot)$ is a submartingale. So using $\vartheta' \in \mathcal{A}_{k-1}$, we get

$$V_{k-1}(x) = V_{k-1}(x+(\vartheta'\bullet X)_{k-1}) \le E[V_k(x+(\vartheta'\bullet X)_k))|\mathcal{F}_{k-1}] = E[V_k(x+\vartheta'_k \triangle X_k)|\mathcal{F}_{k-1}].$$

Taking ess inf yields

$$V_{k-1}(x) \le \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_{k-1}} E[V_k(x + \vartheta'_k \triangle X_k)|\mathcal{F}_{k-1}].$$

To show "$\ge$", we fix $\vartheta \in \mathcal{A}_{k-1}(0)$ and then compute

$$\begin{aligned} E[V_k(x + \vartheta_k \triangle X_k)|\mathcal{F}_{k-1}] &\le E\Big[E\Big[\Big(H - (x + \vartheta_k \triangle X_k) - \sum_{j=k+1}^{T} \vartheta_j \triangle X_j\Big)^2 \Big| \mathcal{F}_k\Big]\Big|\mathcal{F}_{k-1}\Big] \\ &= E\Big[\Big(H - x - \sum_{j=k}^{T} \vartheta_j \triangle X_j\Big)^2 \Big| \mathcal{F}_{k-1}\Big], \end{aligned}$$

where the inequality is obtained by observing that the strategy given by $\tilde{\vartheta}_j = 0$ for $j \le k$ and $\tilde{\vartheta}_j = \vartheta_j$ is in $\mathcal{A}_k(0)$. Taking ess inf on both sides, we get

$$\begin{aligned} &\operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_{k-1}} E[V_k(x + \vartheta_k \triangle X_k)|\mathcal{F}_{k-1}] \\ &\qquad \le \operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_{k-1}} E\Big[\Big(H - x - \sum_{j=k}^{T} \vartheta_j \triangle X_j\Big)^2 \Big| \mathcal{F}_{k-1}\Big] = V_{k-1}(x). \end{aligned}$$

Finally $V_T(x) = (H - x)^2$ is clear by definition of $V_T(x)$.

Exercise 10.2 (Mean-variance hedging continued).

(a) Prove by backward induction that

$$V_k(x) = A_k x^2 + 2B_k x + C_k,$$

where A_k, B_k, C_k are $\mathcal{F}_k$-measurable random variables with $0 \leq A_k \leq 1$ and $A_T = 1, B_T = -H, C_T = H^2$.

(b) Use the Dynamic Programming Principle to construct a candidate for an optimal strategy ϑ^*.

Solution 10.2

(a) *Base:* For $k = T$, we have $V_T(x) = (H - x)^2 = x^2 - 2Hx + H^2$. So $A_T = 1, B_T = -H$, and $C_T = H^2$.

Induction step: Suppose that $V_k(x) = A_k x^2 + 2B_k x + C_k$ with $0 \leq A_k \leq 1$. By part (d) in the previous exercise, we need to compute

$$\begin{aligned}
\operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_{k-1}} E\left[V_k(x + \vartheta_k \triangle X_k) \,|\, \mathcal{F}_{k-1}\right] &= \operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_{k-1}} E[A_k(x + \vartheta_k \triangle X_k)^2 \\
&\quad + 2B_k(x + \vartheta_k \triangle X_k) + C_k|\mathcal{F}_{k-1}] \\
&= \operatorname*{ess\,inf}_{\vartheta \in \mathcal{A}_{k-1}} \{E[A_k x^2 + 2B_k x + C_k|\mathcal{F}_{k-1}] \\
&\quad + 2\vartheta_k E[xA_k \triangle X_k + B_k \triangle X_k|\mathcal{F}_{k-1}] \\
&\quad + \vartheta_k^2 E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]\}.
\end{aligned}$$

This is optimisation of a quadratic polynomial and it depends on whether the leading coefficient is 0 or not.

On the event $G_k := \{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}] = 0\}$, we first observe by the Cauchy-Schwarz inequality for conditional expectations that

$$E[A_k \triangle X_k|\mathcal{F}_{k-1}]^2 = E[\sqrt{A_k}\sqrt{A_k} \triangle X_k|\mathcal{F}_{k-1}]^2 \leq E[A_k|\mathcal{F}_{k-1}]E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}] = 0.$$

On the other hand, note that $B_k^2 \leq A_k C_k$ because $V_k(x) \geq 0$. This implies $\{A_k = 0\} \subseteq \{B_k = 0\}$. We have

$$E[A_k(\triangle X_k)^2 \mathbb{1}_{G_k}] = E[E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]\mathbb{1}_{G_k}] = 0.$$

Using $A_k(\triangle X_k)^2 \mathbb{1}_{G_k} \geq 0$ P-a.s., we obtain $A_k(\triangle X_k)^2 \mathbb{1}_{G_k} = 0$ P-a.s. Thus $B_k(\triangle X_k)^2 \mathbb{1}_{G_k} = 0$ P-a.s. and hence $B_k \triangle X_k \mathbb{1}_{G_k} = 0$ P-a.s. This yields $E[B_k \triangle X_k|\mathcal{F}_{k-1}]\mathbb{1}_{G_k} = 0$ P-a.s. To sum up, we obtain the implication

$$E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}] = 0 \implies E[A_k \triangle X_k|\mathcal{F}_{k-1}] = 0 \text{ and } E[B_k \triangle X_k|\mathcal{F}_{k-1}] = 0.$$

Now the optimisation problem on G_k becomes

$$\begin{aligned}V_{k-1}(x) &= \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_{k-1}} E\left[V_k(x+\vartheta'_k \triangle X_k) \,\middle|\, \mathcal{F}_{k-1}\right]\\ &= \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_{k-1}} E[A_k x^2 + 2B_k x + C_k | \mathcal{F}_{k-1}]\\ &= E[A_k x^2 + 2B_k x + C_k | \mathcal{F}_{k-1}].\end{aligned}$$

Thus $V_{k-1}(x) = A_{k-1}x^2 + 2B_{k-1}x + C_{k-1}$ with $A_{k-1} = E[A_k|\mathcal{F}_{k-1}], B_{k-1} = E[B_k|\mathcal{F}_{k-1}], C_{k-1} = E[C_k|\mathcal{F}_{k-1}]$. This yields $0 \le A_{k-1} \le 1$ and verifies the induction step.

On $G_k^c = \{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}] \neq 0\}$, the optimiser is

$$\vartheta_k(x) = -\frac{E[(xA_k + B_k)\triangle X_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}.$$

Setting $0/0 := 0$, we make $\vartheta_k(x)$ well defined on both G_k and G_k^c. Now substituting $\vartheta_k(x)$ in the above gives

$$\begin{aligned}V_{k-1}(x) &= E[A_k x^2 + 2B_k x + C_k|\mathcal{F}_{k-1}] - 2\frac{(E[(xA_k + B_k)\triangle X_k|\mathcal{F}_{k-1}])^2}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\\ &\quad + \left(\frac{E[(xA_k + B_k)\triangle X_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\right)^2 E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]\\ &= E[A_k x^2 + 2B_k x + C_k|\mathcal{F}_{k-1}] - \frac{(E[(xA_k + B_k)\triangle X_k|\mathcal{F}_{k-1}])^2}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\\ &= x^2\left(E[A_k|\mathcal{F}_{k-1}] - \frac{(E[A_k\triangle X_k|\mathcal{F}_{k-1}])^2}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\right)\\ &\quad + x\left(2E[B_k|\mathcal{F}_{k-1}] - 2\frac{E[A_k\triangle X_k|\mathcal{F}_{k-1}]E[B_k\triangle X_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\right)\\ &\quad + \left(E[C_k|\mathcal{F}_{k-1}] - \frac{(E[B_k\triangle X_k|\mathcal{F}_{k-1}])^2}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\right).\end{aligned}$$

So set

$$\begin{aligned}A_{k-1} &:= E[A_k|\mathcal{F}_{k-1}] - \frac{(E[A_k\triangle X_k|\mathcal{F}_{k-1}])^2}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\\ B_{k-1} &:= E[B_k|\mathcal{F}_{k-1}] - \frac{E[A_k\triangle X_k|\mathcal{F}_{k-1}]E[B_k\triangle X_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}\\ C_{k-1} &:= E[C_k|\mathcal{F}_{k-1}] - \frac{(E[B_k\triangle X_k|\mathcal{F}_{k-1}])^2}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}.\end{aligned}$$

We then have in general

$$V_{k-1}(x) = A_{k-1}x^2 + 2B_{k-1}x + C_{k-1},$$

as well as $0 \le A_{k-1} \le 1$ which proves the induction step.

(b) Note by Dynamic Programming Principle (Exercise 10.1(d)) and part (a), we have

$$\begin{aligned} V_{k-1}(v_{k-1}) &= \operatorname*{ess\,inf}_{\vartheta_k} E[V_k(v_{k-1}+\vartheta_k \triangle X_k)|\mathcal{F}_{k-1}] \\ &= \operatorname*{ess\,inf}_{\vartheta_k} E[A_k(v_{k-1}+\vartheta_k \triangle X_k)^2 + 2B_k(v_{k-1}+\vartheta_k \triangle X_k) + C_k|\mathcal{F}_{k-1}]. \end{aligned}$$

Setting the differential w.r.t ϑ_k to 0, we see that the first order condition is

$$2E[A_k(v_{k-1}+\vartheta_k \triangle X_k)\triangle X_k|\mathcal{F}_{k-1}] + 2E[B_k \triangle X_k|\mathcal{F}_{k-1}] = 0.$$

Using measurability of A_k, B_k, C_k and predictability of the strategy ϑ we get that the optimal ϑ_k for a given v_{k-1} is

$$\vartheta_k^*(v_{k-1}) = -\frac{E[B_k \triangle X_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]} - \frac{E[A_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]} v_{k-1}$$

Finally since $v_{k-1} = c + (\vartheta^* \bullet X)_{k-1}$, we have

$$\vartheta_k^* := -\frac{E[B_k \triangle X_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]} - \frac{E[A_k|\mathcal{F}_{k-1}]}{E[A_k(\triangle X_k)^2|\mathcal{F}_{k-1}]}(c + (\vartheta^* \bullet X)_{k-1}).$$

Thus ϑ_k^*, $k = 1, \dots, T$ give a candidate for an optimal strategy.