Introduction to Mathematical Finance

Exercise sheet 10

Exercise 10.1 (Mean-variance hedging). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Suppose that the discounted price process X satisfies $E[(\triangle X_k)^2|\mathcal{F}_{k-1}] < \infty$ P-a.s. for all k. Define

$$\mathcal{A} := \Big\{\text{all predictable processes } \vartheta = (\vartheta_k)_{k=1,\dots,T} : (\vartheta \bullet X)_k \in L^2 \text{ for } k = 1, \dots, T\Big\}.$$

Let $c \in \mathbb{R}$ and $H \in L^2(\mathcal{F}_T)$. Mean-variance hedging (MVH) is the problem of approximating, with minimal mean squared error, a given payoff by the final value of a self-financing trading strategy in a financial market. We thus consider the problem

$$\min_{\vartheta \in \mathcal{A}} E[(H - c - (\vartheta \bullet X)_T)^2] \tag{1}$$

The goal of this exercise is to construct a candidate for the optimal strategy using the MOP. For $\vartheta \in \mathcal{A}$, we set

$$\begin{aligned}
\mathcal{A}_k(\vartheta) &:= \{\vartheta' \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \le k\},\\
\mathcal{A}_k &:= \mathcal{A}_k(0) = \{\vartheta' \in \mathcal{A} : \vartheta'_j = 0 \text{ for } j \le k\}.
\end{aligned}$$

For $v_k \in L^2(\mathcal{F}_k)$, we define

$$\begin{aligned}
\Gamma_k(v_k, \vartheta') &:= E\Bigg[\bigg(H - v_k - \sum_{j=k+1}^{T} \vartheta'_j \triangle X_j\bigg)^2 \Bigg| \mathcal{F}_k\Bigg],\\
V_k(v_k) &:= \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_k} \Gamma_k(v_k, \vartheta').
\end{aligned}$$

(a) Show that for each k and each $v_k \in L^2(\mathcal{F}_k)$, the collection of random variables

$$\Lambda_k(v_k) := \Big\{\Gamma_k(v_k, \vartheta') : \vartheta' \in \mathcal{A}_k(0)\Big\}$$

is closed under taking minima.

(b) Show that for fixed $\vartheta \in \mathcal{A}$, $x \in \mathbb{R}$, the process $(V_k(x + (\vartheta \bullet X)_k))_{k=0,\dots,T}$ is a submartingale. *Hint: Use Corollary 2 from Appendix E*

(c) Show that $\vartheta^* \in \mathcal{A}$ is optimal if and only if the process $(V_k(c + (\vartheta^* \bullet X)_k))_{k=0,\dots,T}$ is a martingale.

(d) Show that (V_k) satisfies the recursion

$$V_{k-1}(x) = \operatorname*{ess\,inf}_{\vartheta' \in \mathcal{A}_{k-1}} E[V_k(x + \vartheta'_k \triangle X_k) | \mathcal{F}_{k-1}]$$

with $V_T(x) = (H - x)^2$.

Exercise 10.2 (Mean-variance hedging continued).

(a) Prove by backward induction that

$$V_k(x) = A_k x^2 + 2B_k x + C_k,$$

where A_k, B_k, C_k are $\mathcal{F}_k$-measurable random variables with $0 \leq A_k \leq 1$ and $A_T = 1, B_T = -H, C_T = H^2$.

(b) Use the Dynamic Programming Principle to construct a candidate for an optimal strategy ϑ^*.