Introduction to Mathematical Finance

Solution sheet 11

Solution 11.1 The dynamic programming principle (Proposition III.2.5) indicates that we should solve

$$\sigma_k(v_k) = \operatorname{ess\,sup}_{\pi_k,\gamma_k}\left(U_c(\gamma_k v_k) + E\left[\sigma_{k+1}\left(v_k(1+\pi_k\cdot(R_{k+1}-1)-\gamma_k)\right)|\mathcal{F}_k\right]\right)$$

by backwards induction to get candidate optimal solution.

- First note that $\sigma_T(v_T) = U_w(v_T) = \log(v_T)$
- Consider $k = T-1$:

$$\begin{aligned}
\sigma_{T-1}(v_{T-1}) &= \operatorname{ess\,sup}_{\pi_{T-1},\gamma_{T-1}}\left(\log(\gamma_{T-1}v_{T-1}) + E\left[\sigma_T\left(v_{T-1}(1+\pi_{T-1}\cdot(R_T-1)-\gamma_{T-1})\right)|\mathcal{F}_{T-1}\right]\right)\\
&= \operatorname{ess\,sup}_{\pi_{T-1},\gamma_{T-1}}\Big(\log(\gamma_{T-1}) + \log(v_{T-1}) + \\
&\qquad\qquad E\left[\log(v_{T-1}) + \log\left(1+\pi_{T-1}\cdot(R_T-1)-\gamma_{T-1}\right)|\mathcal{F}_{T-1}\right]\Big)\\
&= 2\log(v_{T-1}) + \operatorname{ess\,sup}_{\pi_{T-1},\gamma_{T-1}}\left(\log(\gamma_{T-1}) + E\left[\log((1+\pi_{T-1}\cdot(R_T-1)-\gamma_{T-1})|\mathcal{F}_{T-1}\right]\right)\\
&= 2\log(v_{T-1}) + \\
&\quad \operatorname{ess\,sup}_{\pi_{T-1},\gamma_{T-1}}\left(\log(\gamma_{T-1}) + E\left[\log((1+\pi\cdot(R_T-1)-\gamma)\right]_{|\pi=\pi_{T-1},\gamma=\gamma_{T-1}}\right)\\
&= 2\log(v_{T-1}) + C_{T-1}
\end{aligned}$$

where

$$C_{T-1} := \sup_{\pi,\gamma}\left(\log(\gamma) + E\left[\log(1+\pi\cdot(R-1)-\gamma)\right]\right)$$

In the fourth equality above, we have used that R_T is independent of $\mathcal{F}_{T-1}$ and π_{T-1} and γ_{T-1} are $\mathcal{F}_{T-1}$ measurable. The random variable R appearing in the definition of C_{T-1} is distributed identically as the i.i.d random variables R_k.

- For a general $0 \le k \le T-1$, if $\sigma_{k+1}(v_{k+1}) = (T-(k+1)+1)\log(v_{k+1}) + \sum_{j=k+1}^{T-1} C_j$ for some constants C_j, then:

$$\begin{aligned}
\sigma_k(v_k) &= \operatorname{ess\,sup}_{\pi_k,\gamma_k} \Big(\log(\gamma_k v_k) + E\left[\sigma_{k+1}\left(v_k(1+\pi_k\cdot(R_{k+1}-1)-\gamma_k)\right)|\mathcal{F}_k\right]\Big) \\
&= \operatorname{ess\,sup}_{\pi_k,\gamma_k} \Bigg(\log(\gamma_k) + \log(v_k) + \\
&\qquad E\left[(T-(k+1)+1)\{\log(v_k) + \log\left(1+\pi_k\cdot(R_{k+1}-1)-\gamma_k\right)\} + \sum_{j=k+1}^{T-1} C_j|\mathcal{F}_k\right]\Bigg) \\
&= (T-k+1)\log(v_k) + \sum_{j=k+1}^{T-1} C_j \\
&\quad + \operatorname{ess\,sup}_{\pi_k,\gamma_k} \Big(\log(\gamma_k) + (T-k)E\left[\log((1+\pi_k\cdot(R_{k+1}-1)-\gamma_k)\,|\mathcal{F}_k\right]\Big) \\
&= (T-k+1)\log(v_k) + \sum_{j=k+1}^{T-1} C_j \\
&\quad \operatorname{ess\,sup}_{\pi_k,\gamma_k} \Big(\log(\gamma_k) + (T-k)E\left[\log((1+\pi\cdot(R_{k+1}-1)-\gamma)\right]_{|\pi=\pi_k,\gamma=\gamma_k}\Big) \\
&= (T-k+1)\log(v_k) + \sum_{j=k}^{T-1} C_j
\end{aligned}$$

where

$$C_k := \sup_{\pi,\gamma} \Big(\log(\gamma) + (T-k)E\left[\log(1+\pi\cdot(R-1)-\gamma)\right]\Big)$$

This proves by induction that for all $0 \le k \le T-1$,

$$\sigma_k(v_k) = (T-k+1)\log(v_k) + \sum_{j=k}^{T-1} C_j$$

Assume the constants C_k are finite and that there exist maximisers (π_k^*, γ_k^*) (constrained to statisfy $\gamma_k^* \ge 0$ and $1^T\pi_k^* = 1$ for all k). We then expect by the dynamic programming principle (π_k^*, γ_k^*) to be optimal for the original static utility optimization problem (**??**). Indeed the dynamic programming principle (Proposition III.2.5) tells us that for an optimal solution of (**??**), $\sigma_k(v_k)$ computed above is equal to the optimal remaining utility at time k. In particular,

$$E[\sigma_0(v_0)] = \mathcal{U}^*$$

is the optimal value of (**??**). To find the optimal strategy achieving this value $\mathcal{U}^*$, we first calculate backwards $\sigma_k(v_k)$ and the corresponding optimizers $\pi_k^*(v_k)$ and $\gamma_k^*(v_k)$. In general this backward step gives the optimal strategy *as a function of the current wealth* v_k. To get the optimal strategy we then need to iterate forward starting from the initial wealth v_0: $W_0 = v_0$ and if W_k is known, set

$$\begin{aligned}
\pi_k^* &= \pi_k^*(W_k) \\
\gamma_k^* &= \gamma_k^*(W_k) \\
W_{k+1} &= W_k\left(1+\pi_k^*\cdot(R_{k+1}-1)-\gamma_k^*\right)
\end{aligned}$$

However, calculations simplify considerably in our setup, since the optimal γ_k^* and π_k^* do NOT depend on the current wealth v_k (see Remark below).

Remarks

1. Note that the dynamic programming principle only gives a *candidate* solution to (**??**). To confirm the optimality, one should check the sufficient condition of the Martingale Optimality Principle, i.e. show that $J(\vartheta^*, c^*)$ is a true martingale (where recall that $c_k^* = \gamma_k^* W_k^{v_0,\pi^*,\gamma^*}$ and $\vartheta_{k+1}^{*,l} = \frac{\pi_k^{*,l} W_k^{v_0,\pi^*,\gamma^*}}{X_k^l}$).

2. The (candidate) optimal proportions γ_k^*, π_k^* are deterministic. But there is still trading and stochastic consumption involved because
$$c_k^* = \gamma_k^* W_k^{v_0,\pi^*,\gamma^*}$$
and
$$\vartheta_{k+1}^{*,l} = \frac{\pi_k^{*,l} W_k^{v_0,\pi^*,\gamma^*}}{X_k^l}$$
still depend on the randomly evolving wealth and asset values.

Solution 11.2

(a) First note by induction that X_n is $\sigma(X_0, \epsilon_0, ..., \epsilon_{n-1})$ measurable. Indeed it is clearly true for $n = 0$ and by induction, if X_n is $\sigma(X_0, \epsilon_0, ..., \epsilon_{n-1})$ measurable, then $X_n = f_n(X_0, \epsilon_0, ..., \epsilon_{n-1})$ for a certain measurable function f_n and so
$$X_{n+1} = \phi_n(X_n, \epsilon_n) = \phi_n\left(f_n(X_0, \epsilon_0, ..., \epsilon_{n-1}), \epsilon_n\right)$$
is $\sigma(X_0, \epsilon_0, \dots, \epsilon_n)$ measurable.

Since by assumption, ϵ_n is independent of $\sigma(X_0, \epsilon_0, \dots, \epsilon_{n-1})$, we get by the above result that ϵ_n is independent of $\sigma(X_0, \dots, X_n)$. Hence
$$\begin{aligned} E\left[\mathbb{1}_{\{X_{n+1} \in A\}} | \sigma(X_0, \dots, X_n)\right] &= E\left[\mathbb{1}_{\{\phi_n(X_n, \epsilon_n) \in A\}} | \sigma(X_0, \dots, X_n)\right] \\ &= P_n(X_n, A) \end{aligned}$$
where in the last equality we used the following proposition: if X, Y are random variables such that X is $\mathcal{G}$ measurable with respect to some sigma algebra $\mathcal{G}$ and Y is independent of $\mathcal{G}$, then
$$E[f(X, Y)|\mathcal{G}] = E[f(z, Y)]\Big|_{z=X}$$

(b) The equality holds for $f = \mathbb{1}_A$ for any $A \in \mathcal{E}$ by the previous question. By linearity it also holds for simple functions, i.e functions for the form $f = \sum_{k=1}^n a_k \mathbb{1}_{A_k}$. Finally it also holds for positive measurable functions since every such functions is the increasing limit of a sequence of simple functions.

Solution 11.3

(a) The proof is very similar to the solution of the question 11.1.a). Again by induction one can see that X_n and U_n are $\sigma(X_0, \epsilon_0, ..., \epsilon_{n-1})$ measurable. Indeed it is again clearly true for $n = 0$ and by induction, if X_n and U_n are $\sigma(X_0, \epsilon_0, ..., \epsilon_{n-1})$ measurable, then there exists some measurable functions f_n and g_n such that $X_n = f_n(X_0, \epsilon_0, ..., \epsilon_{n-1})$ and $U_n = g_n(X_0, \epsilon_1, \dots, \epsilon_{n-1})$

$$X_{n+1} = \phi_n(X_n, U_n, \epsilon_n) = \phi_n\left(f_n(X_0, \epsilon_0, ..., \epsilon_{n-1}), g_n(X_0, \epsilon_0, ..., \epsilon_{n-1}), \epsilon_n\right)$$

is $\sigma(X_0, \epsilon_0, \dots, \epsilon_n)$ measurable and thus so is U_{n+1} since U is adapted to the natural filtration of X.

Since by assumption, ϵ_n is independent of $\sigma(X_0, \epsilon_0, \dots, \epsilon_{n-1})$, we get by the above result that ϵ_n is independent of $\sigma(X_0, \dots, X_n)$. Hence we have (using the same proposition as in question 11.1.a)),

$$\begin{aligned} E\left[\mathbb{1}_{\{X_{n+1}\in A\}} | \sigma(X_0, \dots, X_n)\right] &= E\left[\mathbb{1}_{\{\phi_n(X_n, U_n, \epsilon_n)\in A\}} | \sigma(X_0, \dots, X_n)\right] \\ &= P_n^{(U_n)}(X_n, A) \end{aligned}$$

(b) The solution follows the same ideas as question 11.1.b). The equality holds for $f = \mathbb{1}_A$ for any $A \in \mathcal{E}$ by the previous question. By linearity it also holds for simple functions, i.e functions for the form $f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$. Finally it also holds for positive measurable function since every such functions is the increasing limit of a sequence of simple functions.

Solution 11.4 As suggested by the hint, we use bakward induction for both questions.

(a) It is clearly true for $n = N$:

$$O_N(X_N) = \gamma(X_N) = E_{x_0}^{\eta^{min}}\left[\gamma(X_N) | \sigma(X_0, \dots, X_N)\right]$$

Now we suppose that the desired property holds for $n + 1$ and show that it still holds for n

$$\begin{aligned}
O_n(X_n) &= c_n(X_n, U_n^{(min)}(X_n)) + (P_n^{(U_n^{(min)}(X_n))} O_{n+1})(X_n) \quad \textit{(by definition)} \\
&= c_n(X_n, U_n^{(min)}(X_n)) + E_{x_0}^{\eta^{min}} \left[O_{n+1}(X_{n+1}) | \sigma(X_0, \dots, X_n) \right] \\
&\quad \textit{(using the result from question 2)} \\
&= c_n(X_n, U_n^{(min)}(X_n)) \\
&\qquad + E_{x_0}^{\eta^{min}} \left[E_{x_0}^{\eta^{min}} \left[\sum_{k=n+1}^{N-1} c_k(X_k, U_k) + \gamma(X_N) | \sigma(X_0, \dots, X_{n+1}) \right] | \sigma(X_0, \dots, X_n) \right] \\
&\qquad \textit{(by induction hypotheses)} \\
&= c_n(X_n, U_n^{(min)}(X_n)) \\
&\qquad + E_{x_0}^{\eta^{min}} \left[\sum_{k=n+1}^{N-1} c_k(X_k, U_k) + \gamma(X_N) | \sigma(X_0, \dots, X_n) \right] \\
&\qquad \textit{(by tower property)} \\
&= E_{x_0}^{\eta^{min}} \left[\sum_{k=n}^{N-1} c_k(X_k, U_k) + \gamma(X_N) | \sigma(X_0, \dots, X_n) \right]
\end{aligned}$$

(b) Fix a strategy η corresponding to controls U. The claimed result is clearly true for $n = N$:

$$O_N(X_N) = \gamma(X_N) = E_{x_0}^{\eta} \left[\gamma(X_N) | \sigma(X_0, \dots, X_N) \right]$$

Now we suppose that the desired property holds for $n+1$ and show that it still holds for n

$$\begin{aligned}
O_n(X_n) &= c_n(X_n, U_n^{(min)}(X_n)) + (P_n^{(U_n^{(min)}(X_n))} O_{n+1})(X_n) \quad \textit{(by definition)} \\
&\leq c_n(X_n, U_n(X_n)) + (P_n^{(U_n(X_n))} O_{n+1})(X_n) \quad \textit{(by definition of optimal strategy)} \\
&= c_n(X_n, U_n(X_n)) + E_{x_0}^{\eta}\left[O_{n+1}(X_{n+1}) | \sigma(X_0, \ldots, X_n)\right] \\
&\quad \textit{(using the result from question 3)} \\
&\leq c_n(X_n, U_n(X_n)) \\
&\qquad + E_{x_0}^{\eta}\left[E_{x_0}^{\eta}\left[\sum_{k=n+1}^{N-1} c_k(X_k, U_k) + \gamma(X_N) | \sigma(X_0, \ldots, X_{n+1})\right] | \sigma(X_0, \ldots, X_n)\right] \\
&\qquad \textit{(by induction hypotheses + result from question (a))} \\
&= c_n(X_n, U_n(X_n)) \\
&\qquad + E_{x_0}^{\eta}\left[\sum_{k=n+1}^{N-1} c_k(X_k, U_k) + \gamma(X_N) | \sigma(X_0, \ldots, X_n)\right] \\
&\qquad \textit{(by tower property)} \\
&= E_{x_0}^{\eta}\left[\sum_{k=n}^{N-1} c_k(X_k, U_k) + \gamma(X_N) | \sigma(X_0, \ldots, X_n)\right]
\end{aligned}$$