

Introduction to Mathematical Finance

Exercise sheet 12

Exercise 12.1 We place ourselves in the setting of Chapter IV.4.

The goal of this exercise is to illustrate why we need to work with the larger set $\mathcal{Z}(y)$ instead of the set of EMMs \mathbb{P} . We construct an example where the infimum

$$\inf_{Q \in \mathbb{P}} E \left[J \left(y \frac{dQ}{dP} \right) \right]$$

is not attained over \mathbb{P} , but only over $\mathcal{Z}(y)$.

Recall that we try to solve

$$\begin{aligned} u(x) &= \sup_{V \in \mathcal{V}(x)} E[U(V_T)] \\ &= \sup_{f \in \mathcal{C}_+(x)} E[U(f)] \end{aligned}$$

where $\mathcal{V}(x) = \{V = x + G(\vartheta) \geq 0 | \vartheta \in \Theta\}$ is the set of wealth processes of all self-financing 0-admissible trading strategies, and $\mathcal{C}_+(x) = \{V_T | V \in \mathcal{V}(x)\} = \mathcal{C}(x) \cap L_+^0$ is the set of corresponding terminal values. In the lectures you have derived the corresponding dual optimization problem. Intuitively we can construct an upper bound on $u(x)$ by noting that Theorem II.7.2 (hedging duality) implies that $f \in L_+^0$ satisfies

$$f \in \mathcal{C}(x) \iff E_Q[f] \leq x \text{ for all } Q \in \mathbb{P}.$$

Thus

$$u(x) = \sup_{f \in L_+^0} \{E[U(f)] | E_Q[f] \leq x \text{ for all } Q \in \mathbb{P}\}.$$

For any $x > 0$, $f \in L_+^0$, $Q \in \mathbb{P}$ such that $E_Q[f] \leq x$, and $y \geq 0$ (which later will be chosen depending on x in order to guarantee no duality gap), we have:

$$\begin{aligned} E[U(f)] &\leq E[U(f)] + y \left(x - E \left[\frac{dQ}{dP} f \right] \right) \\ &= E \left[U(f) - y \frac{dQ}{dP} f \right] + xy \\ &\leq E \left[J \left(y \frac{dQ}{dP} \right) \right] + xy, \end{aligned}$$

where $J(y) := \sup_{x>0} (U(x) - xy)$, $y > 0$, is the Legendre transform of $-U(-\cdot)$. The RHS does not depend on f anymore. Taking the supremum over $f \in \mathcal{C}_+(x)$ and infimum over $Q \in \mathbb{P}$, we get:

$$u(x) \leq \inf_{Q \in \mathbb{P}} E \left[J \left(y \frac{dQ}{dP} \right) \right] + xy$$

This is **almost** the dual problem!

1. It depends on $y > 0$. We will choose y such that strong duality holds (see Chapter IV.6. in the lecture notes on the homepage).
2. The set \mathbb{P} of EMMs has poor closure properties. So we work with a larger set instead. For $y > 0$, we therefore define

$$\mathcal{Z}(y) := \{ \text{set of all nonnegative adapted } Z = (Z_k)_{k=0,1,\dots,T} \text{ with } \\ Z_0 = y \text{ such that } ZV \text{ is a P-supermartingale for all } V \in \mathcal{V}(x) \}.$$

We also define the set

$$\mathcal{D}(y) := \{ h \in L_+^0 \mid h \leq Z_T \text{ for some } Z \in \mathcal{Z}(y) \}.$$

In the lecture, we have shown that the dual problem involves solving the optimization problem

$$\begin{aligned} j(y) &= \inf_{Z \in \mathcal{Z}(y)} E[J(Z)] \\ &= \inf_{f \in \mathcal{D}(y)} E[J(f)]. \end{aligned}$$

As mentioned in the beginning of the question, the goal of this exercise is to illustrate why we need to work with the larger set $\mathcal{Z}(y)$ instead of the set of EMMs \mathbb{P} . We construct an example where the infimum

$$\inf_{Q \in \mathbb{P}} E \left[J \left(y \frac{dQ}{dP} \right) \right]$$

is not attained over \mathbb{P} , but only over $\mathcal{Z}(y)$.

We construct a one-period market defined on a *countable* probability space Ω . Let $(p_n)_{n=0}^\infty$ be a sequence of strictly positive numbers such that $\sum_{n=0}^\infty p_n = 1$ tending sufficiently fast to 0, and let $(x_n)_{n=0}^\infty$ be a sequence of positive real numbers starting at $x_0 = 2$ and also decreasing to 0, but less fast than $(p_n)_{n=0}^\infty$. For example, $p_0 = 1 - \alpha$, $p_n = \alpha 2^{-n}$, for $n \geq 1$, and $x_0 = 2$, $x_n = 1/n$, for $n \geq 1$, will do, if $0 < \alpha < 1$ is small enough to satisfy $(1 - \alpha)/2 + \alpha \sum_{n=0}^\infty 2^{-n}(-n + 1) > 0$. Finally, consider a market with a numéraire and one risky asset S with initial discounted price $X_0 = 1$ and terminal discounted price X_1 taking the values $(x_n)_{n=0}^\infty$ with probabilities $(p_n)_{n=0}^\infty$. We equip the probability space with the natural filtration of S .

- (a) Show that the market is arbitrage free, and argue that $\mathbb{P} \neq \emptyset$. Is the market complete?
- (b) Determine an interval $[a, b]$, with $a < b$ in \mathbb{R} , of the values for ϑ_1 such that $1 + \vartheta_1 \cdot \Delta X_1 \in \mathcal{V}(1)$, i.e. $1 + \vartheta_1 \cdot \Delta X_1 \geq 0$ P-almost surely.
- (c) Maximise the function

$$f(\vartheta_1) := E[\log(1 + \vartheta_1 \Delta X_1)]$$

over $[a, b]$.

Hint: Compute the derivative $f'(\vartheta_1)$ and plug-in the value of b from question b).

Derive the optimal investment $V^* \in \mathcal{V}(1)$.

- (d) Compute explicitly (in terms of $(x_n)_{n=0}^\infty$ and $(p_n)_{n=0}^\infty$) the value function $u(x)$. Show that $u'(1) = 1$.
- (e) Compute the corresponding dual optimizer $Z^* \in \mathcal{Z}(1)$ using $Z_T^*(y) = U'(f_x^*)$ and $y_x^* = u'(1)$ (see Theorem IV.6.2 in the lecture notes on the homepage).
- (f) Conclude that $Z^* \in \mathcal{Z}(1)$ is not a martingale, but only a supermartingale. In particular, Z^* is not the density process of a martingale measure for the process S , and hence the infimum

$$\inf_{Q \in \mathbb{P}} E \left[J \left(y \frac{dQ}{dP} \right) \right]$$

is not attained.

Solution 12.1

- (a) We show that there is no arbitrage of the first kind. Note that a predictable strategy corresponds to a deterministic $\vartheta_1 \in \mathbb{R}$. If ϑ_1 is an arbitrage candidate of the first kind, then $\vartheta_1 \neq 0$, as otherwise $\vartheta_1 \cdot \Delta X_1 > 0$ with positive probability could not hold. If $\vartheta_1 < 0$, then the condition $\vartheta_1(X_1 - X_0) = \vartheta_1(X_1 - 1) \geq 0$ would imply $X_1 \leq 1$ P-almost surely, which is not possible since $x_0 = 2$. Similarly, for $\vartheta_1 > 0$, the arbitrage condition would imply $X_1 \geq 1$ P-almost surely, which again is not possible since $x_n \rightarrow 0$ as $n \rightarrow \infty$. There exists thus no arbitrage of the first kind, and by Proposition I.3.1, the market is arbitrage free. Hence, $\mathbb{P} \neq \emptyset$ by the fundamental theorem of asset pricing.

The market is not complete. Given a contingent claim with payoff H , one has to solve the system of equations $H = v_0 + \vartheta_1 \cdot X_1$. This is a system with a infinitely many equations, but only two unknowns. Hence for general H , the system does not admit a solution. For example, the claim $H = (X_1)^2$ is not replicable.

(b) The condition $1 + \vartheta_1 \cdot \Delta X_1 \geq 0$ P-a.s. is equivalent to the conditions

$$\begin{aligned} 1 + \vartheta_1(2 - 1) &\geq 0 \\ 1 + \vartheta_1\left(\frac{1}{n} - 1\right) &\geq 0 \text{ for all } n \geq 1 \end{aligned}$$

The first equation gives $\vartheta_1 \geq -1$. Taking the limit as $n \rightarrow \infty$ in the second equation, we get $\vartheta_1 \leq 1$. We thus conclude that $1 + \vartheta_1(X_1 - X_0) \in \mathcal{V}(1)$ iff $\vartheta_1 \in [-1, 1]$.

(c) We obtain using elementary calculations

$$f'(\vartheta_1) = \sum_{n=0}^{\infty} p_n \frac{x_n - 1}{1 + \vartheta_1(x_n - 1)}.$$

Therefore $f'(\vartheta_1)$ is strictly positive for $-1 \leq \vartheta_1 \leq 1$ if $\alpha > 0$ the above assumption $f'(1) = (1 - \alpha)/2 + \alpha \sum_{n=0}^{\infty} 2^{-n}(-n + 1) > 0$. Hence $f(\vartheta_1)$ is strictly increasing and attains its maximum on $[-1, 1]$ at $\vartheta_1 = 1$. The optimal investment process $V^* \in \mathcal{V}(1)$ thus equals the process X .

(d) We have already shown that the optimal investment process $V^* \in \mathcal{V}(1)$ equals the process X . We can thus explicitly compute the value process $u(x)$:

$$\begin{aligned} u(x) &= E[U(xX_1)] = \sum_{n=0}^{\infty} p_n(xx_n) \\ &= \sum_{n=0}^{\infty} p_n (\log(x) + \log(x_n)) \\ &= \log(x) + \sum_{n=0}^{\infty} p_n \log(x_n). \end{aligned}$$

In particular, $u'(1) = 1$.

(e) We have already shown that the optimal investment process $V^* \in \mathcal{V}(1)$ equals the process S . By Theorem IV.6.2, the solutions of the primal problem

$$\begin{aligned} u(x) &= \sup_{V \in \mathcal{V}(x)} E[U(V_T)] \\ &= \sup_{f \in \mathcal{C}_+(x)} E[U(f)] \end{aligned}$$

and the dual problem

$$\begin{aligned} j(y) &= \inf_{Z \in \mathcal{Z}(y)} E[J(Z)] \\ &= \inf_{f \in \mathcal{D}(y)} E[J(h)]. \end{aligned}$$

are related by

$$f_x^* = I(h_y^*),$$

where $I = (U')^{-1}$, $y > 0$ is given by the relation $y_z^* = u'(1)$, and $f_x^* \in \mathcal{C}_+(x)$ (respectively $h_y^* \in \mathcal{D}(y)$) is the unique solution to the primal (respectively dual) problem. We thus have

$$h_y^* = I^{-1}(f_x^*) = U'(f_x^*)$$

or equivalently

$$Z^* = U'(V^*) = \frac{1}{X_1}.$$

(f) Note that

$$E[X_1^{-1}] = \sum_{n=0}^{\infty} \frac{p_n}{x_n} = \frac{p_0}{2} + \sum_{n=1}^{\infty} np_n$$

is strictly less than $X_0 = 1$ by the condition $(1-\alpha)/2 + \alpha \sum_{n=0}^{\infty} 2^{-n}(-n+1) > 0$. In particular, the optimal dual $Z^* \in \mathcal{Z}(1)$ is not a martingale (not even a local martingale) but only a supermartingale, and Z^* is therefore not the density of a martingale measure for the process S .

Exercise 12.2 Recall that in the beginning of Chapter IV, we defined the *indirect utility function* (or *value function*) for a given utility function U as

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)]$$

where $\mathcal{V}(x)$ is the set of all wealth processes of self-financing 0-admissible strategies with initial wealth x . Using that U is increasing and concave, show that

- (a) the map $x \mapsto u(x)$ is increasing
- (b) the map $x \mapsto u(x)$ is concave

In this course we made the standing assumption that $u(x_0) < \infty$ for some $x_0 > 0$. Show that under this assumption,

- (c) $u(x) < \infty \quad \forall x > 0$

Solution 12.2

- (a) Fix $0 < x \leq y$. Then for all $\vartheta \in \Theta_{\text{adm}}$, we have $x + G_T(\vartheta) \leq y + G_T(\vartheta)$. So using increasingness of U , we have $U(x + G_T(\vartheta)) \leq U(y + G_T(\vartheta))$. Note also that $\mathcal{V}(x) \subseteq \mathcal{V}(y)$ and hence taking the supremum over ϑ yields that u is increasing.
- (b) Let $\lambda \in (0, 1)$. Note that for $V_T^1 \in \mathcal{V}(x)$, $V_T^2 \in \mathcal{V}(y)$, we have $\lambda V_T^1 \in \mathcal{V}(\lambda x)$ and $(1 - \lambda)V_T^2 \in \mathcal{V}((1 - \lambda)y)$. Moreover, $\lambda V_T^1 + (1 - \lambda)V_T^2$ is in $\mathcal{V}(\lambda x + (1 - \lambda)y)$. Thus

$$\begin{aligned} u(\lambda x + (1 - \lambda)y) &\geq E[U(\lambda V_T^1 + (1 - \lambda)V_T^2)] \\ &\geq \lambda E[U(V_T^1)] + (1 - \lambda)E[U(V_T^2)]. \end{aligned}$$

Taking the supremum over V_T^1 and V_T^2 yields

$$u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y)$$

- (c) First note that $u(y) < \infty$ for all $y \leq x_0$ since $u(\cdot)$ is increasing. Remains to show that $u(y) < \infty$ for all $y > x_0$. Take $x \in (0, x_0)$ and note that since $y > x_0$, we can find some $\lambda \in (0, 1)$ such that

$$x_0 = \lambda x + (1 - \lambda)y$$

Using concavity of u we then have

$$u(x_0) \geq \lambda u(x) + (1 - \lambda)u(y)$$

which implies $u(y) < \infty$ since $u(x_0) < \infty$.

Alternatively, suppose for the sake of finding a contradiction that $u(y) = +\infty$

for some $y > x_0$. By concavity of u , $u(\lambda x_0 + (1 - \lambda)y) = +\infty$ for all $\lambda \in (0, 1)$ and hence $u(z) = +\infty \quad \forall x_0 < z \leq y$ and finally using the increasingness of $u(\cdot)$, $u(z) = +\infty \quad \forall z > x_0$. Let $\epsilon > 0$. Then by the previous observations $u(x_0 - \epsilon) < \infty$ and $u(x_0 + \epsilon) = +\infty$. By concavity of u , the segment joining $u(x_0 - \epsilon)$ and $u(x_0 + \epsilon)$ must lie below the function. Taking the limit as $\epsilon \rightarrow 0$, this concavity property could only hold if $u(x_0) = +\infty$, a contradiction.

Exercise 12.3 Fix an initial wealth x and let $\mathcal{V}(x)$ denote the set of all wealth processes of self-financing 0-admissible strategies with initial wealth x . In Proposition IV.2.1, we proved (in the case of a complete market) the existence of an optimal solution to the terminal wealth utility optimization problem

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)]$$

Show the uniqueness of the optimal solution P -a.s.

Solution 12.3 First note that by Lemma IV.1.2, the wealth process (though maybe not the strategy) is completely determined by its final value. It is thus enough to show the uniqueness of final values to deduce the uniqueness of the optimal wealth process as well.

Suppose to the contrary that V^* and V^{**} are two optimal solutions with P -as. different final values (i.e. $P[V_T^* \neq V_T^{**}] > 0$). Write $u^*(x) = E[U(V_T^*)] = E[U(V_T^{**})]$. Consider the convex combination $\tilde{V} = \frac{1}{2}(V^* + V^{**})$. Note that $\mathcal{V}(x)$ is convex and hence \tilde{V} belongs to the set of feasible solutions. Indeed if ϑ and ϑ' are 0-admissible self-financing strategies with initial wealth x , then so is any convex combination of ϑ and ϑ' and hence

$$V^{conv} = \lambda(x + G(\vartheta)) + (1 - \lambda)(x + G(\vartheta')) = x + G(\lambda\vartheta + (1 - \lambda)\vartheta')$$

is an element of $\mathcal{V}(x)$.

Moreover, using *strict* concavity of the utility function U and $P[V_T^* \neq V_T^{**}] > 0$, we have

$$U(\tilde{V}_T) = U\left(\frac{1}{2}(V_T^* + V_T^{**})\right) > \frac{1}{2}U(V_T^*) + \frac{1}{2}U(V_T^{**})$$

Taking expectation on both sides we get

$$E[U(\tilde{V}_T)] > \frac{1}{2}E[U(V_T^*)] + \frac{1}{2}E[U(V_T^{**})] = u^*(x)$$

contracting the optimality of V^* . The argument here shows that if V^* and V^{**} are two optimal solutions with different FINAL VALUES, then we get a contradiction. This thus gives the uniqueness of final values. Uniqueness of the wealth process then follows from Lemma IV.1.2.

Remark: Important to distinguish between wealth processes and their final values.

Exercise 12.4 Suppose that the utility function U is in C^2 and denote by J its conjugate. Show that $J' = -(U')^{-1}$ and J is strictly convex and lower semicontinuous. Show also that $J'(0) = -\infty$, $J'(\infty) = 0$.

Solution 12.4 By definition, $J(y) := \sup_{x>0}(U(x) - xy)$. So by the first order condition, the supremum is attained for $U'(x) - y = 0$ i.e. at $x = (U')^{-1}(y) =: I(y)$. Therefore, we may write $J(y) = U(I(y)) - I(y)y$. Because $U \in C^2$, the RHS is continuously differentiable. So

$$\begin{aligned} J'(y) &= U'(I(y))I'(y) - I'(y)y - I(y) = yI'(y) - I'(y)y - I(y) = -I(y) \\ &= -(U')^{-1}(y). \end{aligned}$$

For any $x > 0$, the function $y \mapsto U(x) - xy$ is affine hence convex and lower-semicontinuous. Its epigraph (i.e. $\{(y, t) \in \mathbb{R} \times \mathbb{R} : U(x) - xy \leq t\}$) is thus closed and convex. Then, $J(y)$ is the pointwise supremum of the above functions, and its epigraph is the intersection of the above affine functions' epigraphs. Each are closed and convex, proving the epigraph of $J(y)$ is closed and convex, thus $J(y)$ is convex and lower semicontinuous.

Finally, $J'(0) = -\infty$, $J'(\infty) = 0$ follow from the Inada conditions.