Introduction to Mathematical Finance

Exercise sheet 12

Exercise 12.1 We place ourselves in the setting of Chapter IV.4.
The goal of this exercise is to illustrate why we need to work with the larger set $\mathcal{Z}(y)$ *instead of the set of EMMs* $\mathbb{P}$. *We construct an example where the infinum*

$$\inf_{Q\in\mathbb{P}} E\left[J\left(y\frac{dQ}{dP}\right)\right]$$

is not attained over $\mathbb{P}$, *but only over* $\mathcal{Z}(y)$.
Recall that we try to solve

$$\begin{aligned} u(x) &= \sup_{V\in\mathcal{V}(x)} E[U(V_T)] \\ &= \sup_{f\in\mathcal{C}_+(x)} E[U(f)] \end{aligned}$$

where $\mathcal{V}(x) = \{V = x + G(\vartheta) \geq 0 | \vartheta \in \Theta\}$ is the set of wealth processes of all self-financing 0-admissible trading strategies, and $\mathcal{C}_+(x) = \{V_T | V \in \mathcal{V}(x)\} = \mathcal{C}(x) \cap L^0_+$ is the set of corresponding terminal values. In the lectures you have derived the corresponding dual optimization problem. Intuitively we can construct an upper bound on $u(x)$ by noting that Theorem II.7.2 (hedging duality) implies that $f \in L^0_+$ satisfies

$$f \in \mathcal{C}(x) \iff E_Q[f] \leq x \text{ for all } Q \in \mathbb{P}.$$

Thus

$$u(x) = \sup_{f\in L^0_+} \{E[U(f)] | E_Q[f] \leq x \text{ for all } Q \in \mathbb{P}\}.$$

For any $x > 0$, $f \in L^0_+$, $Q \in \mathbb{P}$ such that $E_Q[f] \leq x$, and $y \geq 0$ (which later will be chosen depending on x in order to guarantee no duality gap), we have:

$$\begin{aligned} E[U(f)] &\leq E[U(f)] + y\left(x - E\left[\frac{dQ}{dP}f\right]\right) \\ &= E\left[U(f) - y\frac{dQ}{dP}f\right] + xy \\ &\leq E\left[J\left(y\frac{dQ}{dP}\right)\right] + xy, \end{aligned}$$

where $J(y) := \sup_{x>0} (U(x) - xy)$, $y > 0$, is the Legendre transform of $-U(-\cdot)$. The RHS does not depend on f anymore. Taking the supremum over $f \in \mathcal{C}_+(x)$ and infimum over $Q \in \mathbb{P}$, we get:

$$u(x) \leq \inf_{Q \in \mathbb{P}} E\left[J\left(y\frac{dQ}{dP}\right)\right] + xy$$

This is **almost** the dual problem!

1. It depends on $y > 0$. We will choose y such that strong duality holds (see Chapter IV.6. in the lecture notes on the homepage).

2. The set $\mathbb{P}$ of EMMs has poor closure properties. So we work with a larger set instead. For $y > 0$, we therefore define

$$\begin{aligned}\mathcal{Z}(y) := \{ &\text{ set of all nonnegative adapted} Z = (Z_k)_{k=0,1,\cdots,T} \text{ with} \\ &Z_0 = y \text{ such that } ZV \text{ is a P-supermartingale for all } V \in \mathcal{V}(x)\}.\end{aligned}$$

   We also define the set

$$\mathcal{D}(y) := \{h \in L^0_+ | h \leq Z_T \text{ for some } Z \in \mathcal{Z}(y)\}.$$

   In the lecture, we have shown that the dual problem involves solving the optimization problem

$$\begin{aligned} j(y) &= \inf_{Z \in \mathcal{Z}(y)} E[J(Z)] \\ &= \inf_{f \in \mathcal{D}(y)} E[J(h)]. \end{aligned}$$

As mentioned in the beginning of the question, the goal of this exercise is to illustrate why we need to work with the larger set $\mathcal{Z}(y)$ instead of the set of EMMs $\mathbb{P}$. We construct an example where the infinum

$$\inf_{Q \in \mathbb{P}} E\left[J\left(y\frac{dQ}{dP}\right)\right]$$

is not attained over $\mathbb{P}$, but only over $\mathcal{Z}(y)$.

We construct a one-period market defined on a *countable* probability space Ω. Let $(p_n)_{n=0}^\infty$ be a sequence of strictly positive numbers such that $\sum_{n=0}^\infty p_n = 1$ tending sufficiently fast to 0, and let $(x_n)_{n=0}^\infty$ be a sequence of positive real numbers starting at $x_0 = 2$ and also decreasing to 0, but less fast than $(p_n)_{n=0}^\infty$. For example, $p_0 = 1 - \alpha$, $p_n = \alpha 2^{-n}$, for $n \geq 1$, and $x_0 = 2$, $x_n = 1/n$, for $n \geq 1$, will do, if $0 < \alpha < 1$ is small enough to satisfy $(1-\alpha)/2 + \alpha \sum_{n=0}^\infty 2^{-n}(-n+1) > 0$. Finally, consider a market with a numéraire and one risky asset S with initial discounted price $X_0 = 1$ and terminal discounted price X_1 taking the values $(x_n)_{n=0}^\infty$ with probabilities $(p_n)_{n=0}^\infty$. We equip the probability space with the natural filtration of S.

(a) Show that the market is arbitrage free, and argue that $\mathbb{P} \neq \emptyset$. Is the market complete?

(b) Determine an interval $[a, b]$, with $a < b$ in $\mathbb{R}$, of the values for ϑ_1 such that $1 + \vartheta_1 \cdot \Delta X_1 \in \mathcal{V}(1)$, i.e $1 + \vartheta_1 \cdot \Delta X_1 \geq 0$ P-almost surely.

(c) Maximise the function

$$f(\vartheta_1) := E[\log(1 + \vartheta_1 \Delta X_1)]$$

over $[a, b]$.
Hint: Compute the derivative $f'(\vartheta_1)$ and plug-in the value of b form question b).
Derive the optimal investment $V^* \in \mathcal{V}(1)$.

(d) Compute explicitly (in terms of $(x_n)_{n=0}^{\infty}$ and $(p_n)_{n=0}^{\infty}$) the value function $u(x)$. Show that $u'(1) = 1$.

(e) Compute the corresponding dual optimizer $Z^* \in \mathcal{Z}(1)$ using $Z_T^*(y) = U'(f_x^*)$ and $y_x^* = u'(1)$ (see Theorem IV.6.2 in the lecture notes on the homepage).

(f) Conclude that $Z^* \in \mathcal{Z}(1)$ is not a martingale, but only a supermartingale. In particular, Z^* is not the density process of a martingale measure for the process S, and hence the infinum

$$\inf_{Q \in \mathbb{P}} E\left[J\left(y\frac{dQ}{dP}\right)\right]$$

is not attained.

Exercise 12.2 Recall that in the beginning of Chapter IV, we defined the *indirect utility function (or value function)* for a given utility function U as

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)]$$

where $\mathcal{V}(x)$ is the set of all wealth processes of self-financing 0-admissible strategies with initial wealth x. Using that U is increasing and concave, show that

(a) the map $x \mapsto u(x)$ is increasing

(b) the map $x \mapsto u(x)$ is concave

In this course we made the standing assumption that $u(x_0) < \infty$ for some $x_0 > 0$. Show that under this assumption,

(c) $u(x) < \infty \quad \forall x > 0$

Exercise 12.3 Fix an initial wealth x and let $\mathcal{V}(x)$ denote the set of all wealth processes of self-financing 0-admissible strategies with initial wealth x. In Proposition IV.2.1, we proved (in the case of a complete market) the existence of an optimal solution to the terminal wealth utility optimization problem

$$u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)]$$

Show the uniqueness of the optimal solution P-a.s.

Exercise 12.4 Suppose that the utility function U is in C^2 and denote by J its conjugate. Show that $J' = -(U')^{-1}$ and J is strictly convex and lower semicontinuous. Show also that $J'(0) = -\infty$, $J'(\infty) = 0$.