Introduction to Mathematical Finance

Solution sheet 12

Solution 12.1

(a) We show that there is no arbitrage of the first kind. Note that a predictable strategy corresponds to a deterministic $\vartheta_1 \in \mathbb{R}$. If ϑ_1 is an arbitrage candidate of the first kind, then $\vartheta_1 \neq 0$, as otherwise $\vartheta_1 \cdot \Delta X_1 > 0$ with positive probability could not hold. If $\vartheta_1 < 0$, then the condition $\vartheta_1(X_1 - X_0) = \vartheta_1(X_1 - 1) \geq 0$ would imply $X_1 \leq 1$ P-almost surely, which is not possible since $x_0 = 2$. Similarly, for $\vartheta_1 > 0$, the arbitrage condition would imply $X_1 \geq 1$ P-almost surely, which again is not possible since $x_n \to 0$ as $n \to \infty$. There exists thus no arbitrage of the first kind, and by Proposition I.3.1, the market is arbitrage free. Hence, $\mathbb{P} \neq \emptyset$ by the fundamental theorem of asset pricing.

The market is not complete. Given a contingent claim with payoff H, one has to solve the system of equations $H = v_0 + \vartheta_1 \cdot X_1$. This is a system with a infinitely many equations, but only two unknowns. Hence for general H, the system does not admit a solution. For example, the claim $H = (X_1)^2$ is not replicable.

(b) The condition $1 + \vartheta_1 \cdot \Delta X_1 \geq 0$ P-a.s. is equivalent to the conditions

$$1 + \vartheta_1(2-1) \geq 0$$
$$1 + \vartheta_1(\frac{1}{n} - 1) \geq 0 \text{ for all } n \geq 1$$

The first equation gives $\vartheta_1 \geq -1$. Taking the limit as $n \to \infty$ in the second equation, we get $\vartheta_1 \leq 1$. We thus conclude that $1 + \vartheta_1(X_1 - X_0) \in \mathcal{V}(1)$ iff $\vartheta_1 \in [-1, 1]$.

(c) We obtain using elementary calculations

$$f'(\vartheta_1) = \sum_{n=0}^{\infty} p_n \frac{x_n - 1}{1 + \vartheta_1(x_n - 1)}.$$

Therefore $f'(\vartheta_1)$ is strictly positive for $-1 \leq \vartheta_1 \leq 1$ if $\alpha > 0$ the above assumption $f'(1) = (1-\alpha)/2 + \alpha \sum_{n=0}^{\infty} 2^{-n}(-n+1) > 0$. Hence $f(\vartheta_1)$ is strictly increasing and attains its maximum on $[-1, 1]$ at $\vartheta_1 = 1$. The optimal investment process $V^* \in \mathcal{V}(1)$ thus equals the process X.

(d) We have already shown that the optimal investment process $V^* \in \mathcal{V}(1)$ equals the process X. We can thus explicitly compute the value process $u(x)$:

$$\begin{aligned} u(x) = E[U(xX_1)] &= \sum_{n=0}^{\infty} p_n(xx_n) \\ &= \sum_{n=0}^{\infty} p_n\left(\log(x) + \log(x_n)\right) \\ &= \log(x) + \sum_{n=0}^{\infty} p_n \log(x_n). \end{aligned}$$

In particular, $u'(1) = 1$.

(e) We have already shown that the optimal investment process $V^* \in \mathcal{V}(1)$ equals the process S. By Theorem IV.6.2, the solutions of the primal problem

$$\begin{aligned} u(x) &= \sup_{V \in \mathcal{V}(x)} E[U(V_T)] \\ &= \sup_{f \in \mathcal{C}_+(x)} E[U(f)] \end{aligned}$$

and the dual problem

$$\begin{aligned} j(y) &= \inf_{Z \in \mathcal{Z}(y)} E[J(Z)] \\ &= \inf_{f \in \mathcal{D}(y)} E[J(h)]. \end{aligned}$$

are related by

$$f_x^* = I(h_y^*),$$

where $I = (U')^{-1}$, $y > 0$ is given by the relation $y_z^* = u'(1)$, and $f_x^* \in \mathcal{C}_+(x)$ (receptively $h_y^* \in \mathcal{D}(y)$) is the unique solution to the primal (respectively dual) problem. We thus have

$$h_y^* = I^{-1}(f_x^*) = U'(f_x^*)$$

or equivalently

$$Z^* = U'(V^*) = \frac{1}{X_1}.$$

(f) Note that

$$E[X_1^{-1}] = \sum_{n=0}^{\infty} \frac{p_n}{x_n} = \frac{p_0}{2} + \sum_{n=1}^{\infty} np_n$$

is strictly less than $X_0 = 1$ by the condition $(1-\alpha)/2 + \alpha \sum_{n=0}^{\infty} 2^{-n}(-n+1) > 0$. In particular, the optimal dual $Z^* \in \mathcal{Z}(1)$ is not a martingale (not even a local martingale) but only a supermartingale, and Z^* is therefore not the density of a martingale measure for the process S.

Solution 12.2

(a) Fix $0 < x \leq y$. Then for all $\vartheta \in \Theta_{\mathrm{adm}}$, we have $x + G_T(\vartheta) \leq x + G_T(\vartheta)$. So using increasingness of U, we have $U(x + G_T(\vartheta)) \leq U(y + G_T(\vartheta))$. Note also that $\mathcal{V}(x) \subseteq \mathcal{V}(y)$ and hence taking the supremum over ϑ yields that u is increasing.

(b) Let $\lambda \in (0,1)$. Note that for $V_T^1 \in \mathcal{V}(x), V_T^2 \in \mathcal{V}(y)$, we have $\lambda V_T^1 \in \mathcal{V}(\lambda x)$ and $(1-\lambda)V_T^2 \in \mathcal{V}((1-\lambda)y)$. Moreover, $\lambda V_T^1 + (1-\lambda)V_T^2$ is in $\mathcal{V}(\lambda x + (1-\lambda y)$. Thus

$$\begin{aligned} u\Big(\lambda x + (1-\lambda)y\Big) &\geq E[U(\lambda V_T^1 + (1-\lambda)V_T^2)] \\ &\geq \lambda E[U(V_T^1)] + (1-\lambda)E[U(V_T^2)]. \end{aligned}$$

Taking the supremum over V_T^1 and V_T^2 yields

$$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$$

(c) First note that $u(y) < \infty$ for all $y \leq x_0$ since $u(\cdot)$ is increasing. Remains to show that $u(y) < \infty$ for all $y > x_0$. Take $x \in (0, x_0)$ and note that since $y > x_0$, we can find some $\lambda \in (0,1)$ such that

$$x_0 = \lambda x + (1-\lambda)y$$

Using concavity of u we then have

$$u(x_0) \geq \lambda u(x) + (1-\lambda)u(y)$$

which implies $u(y) < \infty$ since $u(x_0) < \infty$.
Alternatively, suppose for the sake of finding a contradiction that $u(y) = +\infty$ for some $y > x_0$. By concavity of u, $u(\lambda x_0 + (1-\lambda)y) = +\infty$ for all $\lambda \in (0,1)$ and hence $u(z) = +\infty \quad \forall x_0 < z \leq y$ and finally using the increasingness of $u(\cdot)$, $u(z) = +\infty \quad \forall z > x_0$. Let $\epsilon > 0$. Then by the previous observations $u(x_0 - \epsilon) < \infty$ and $u(x_0 + \epsilon) = +\infty$. By concavity of u, the segment joining $u(x_0 - \epsilon)$ and $u(x_0 + \epsilon)$ must lie below the function. Taking the limit as $\epsilon \to 0$, this concavity property could only hold if $u(x_0) = +\infty$, a contradiction.

Solution 12.3 First note that by Lemma IV.1.2, the wealth process (though maybe not the strategy) is completely determined by its final value. It is thus enough to show the uniquesness of final values to deduce the uniqueness of the optimal wealth process as well.
Suppose to the contrary that V^* and V^{**} are two optimal solutions with P-as. different final values (i.e. $P[V_T^* \neq V_T^{**}] > 0$). Write $u^*(x) = E[U(V_T^*)] = E[U(V_T^{**})]$. Consider the convex combination $\tilde{V}_{\cdot} = \frac{1}{2}(V_{\cdot}^* + V_{\cdot}^{**})$. Note that $\mathcal{V}(x)$ is convex and hence $\tilde{V}$ belongs to the set of feasible solutions. Indeed if ϑ and ϑ' are 0-admissible

self-financing strategies with initial wealth x, then so is any convex combination of ϑ and ϑ' and hence

$$V_\cdot^{conv} = \lambda(x + G_\cdot(\vartheta)) + (1-\lambda)(x + G_\cdot(\vartheta')) = x + G_\cdot(\lambda\vartheta + (1-\lambda)\vartheta')$$

is an element of $\mathcal{V}(x)$.
Moreover, using *strict* concavity of the utility function U and $P[V_T^* \neq V_T^{**}] > 0$, we have

$$U(\tilde{V}_T) = U\left(\frac{1}{2}(V_T^* + V_T^{**})\right) > \frac{1}{2}U(V_T^*) + \frac{1}{2}U(V_T^{**})$$

Taking expecation on both sides we get

$$E\left[U(\tilde{V}_T)\right] > \frac{1}{2}E[U(V^*)] + \frac{1}{2}E[U(V^{**})] = u^*(x)$$

contracting the optimality of V^*. The argument here shows that if V^* and V^{**} are two optimal solutions with different FINAL VALUES, then we get a contradiction. This thus gives the uniqueness of final values. Uniqueness of the wealth process then follows from Lemma IV.1.2.

Remark: Important to distinguish between wealth processes and their final values.

Solution 12.4 By definition, $J(y) := \sup_{x>0}(U(x) - xy)$. So by the first order condition, the supremum is attained for $U'(x) - y = 0$ i.e. at $x = (U')^{-1}(y) =: I(y)$. Therefore, we may write $J(y) = U(I(y)) - I(y)y$. Because $U \in C^2$, the RHS is continuously differentiable. So

$$\begin{aligned} J'(y) &= U'(I(y))I'(y) - I'(y)y - I(y) = yI'(y) - I'(y)y - I(y) = -I(y) \\ &= -(U')^{-1}(y). \end{aligned}$$

For any $x > 0$, the function $y \mapsto U(x) - xy$ is affine hence convex and lower-semicontinuous. It's epigraph (i.e $\{(y,t) \in \mathbb{R} \times \mathbb{R} : U(x) - xy \leq t\}$) is thus closed and convex. Then, $J(y)$ is the pointwise supremum of the above functions, and its epigraph is the intersection of the above affine functions' epigraphs. Each are closed and convex, proving the epigraph of $J(y)$ is closed and convex, thus $J(y)$ is convex and lower semicontinuous.
Finally, $J'(0) = -\infty$, $J'(\infty) = 0$ follow from the Inada conditions.