ETH Zürich, FS 2020
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Introduction to Mathematical Finance

Exercise sheet 1

Please hand in your solutions until Friday, 28/02/2020, 13:00 into Bálint Gersey's box next to HG G 53.2.

Exercise 1.1 Let $\mathcal{C} := \mathbb{R} \times \mathbb{R}^K$ be the consumption space with the payoff matrix $\mathcal{D}$ and let e^i, π be an endowment, and a price vector, respectively. Recall the budget set

$$B(e^i, \pi) := \{c \in \mathcal{C} : \exists \vartheta \in \mathbb{R}^N \text{ with } c_0 \leq e^i_0 - \vartheta \cdot \pi \text{ and } c_T \leq e^i_T + \mathcal{D}\vartheta\}.$$

(a) Show $c \in B(e^i, \pi) \iff c - e^i \in B(0, \pi) \Longleftarrow c - e^i$ is attainable with 0 initial wealth.

(b) Show by an example that the converse of the second implication is not true in general.

Exercise 1.2

(a) Construct a market with arbitrage of the first kind but with no arbitrage of the second kind.

(b) Construct a market with arbitrage of the second kind but with no arbitrage of the first kind.

(c) Prove Proposition I.3.1. That is suppose there exists an asset $\mathcal{D}^l$ with $\mathcal{D}^l \geq 0$ and $\mathcal{D}^l \not\equiv 0$. Show that under this assumption, the market is arbitrage-free iff there is no arbitrage of first kind.

Exercise 1.3 Let $\succeq$ be a preference order on $\mathcal{C}$ satisfying axioms (P1)-(P5). A function $\mathcal{U} : \mathcal{C} \to \mathbb{R}$ is called a *utility functional representing* $\succeq$ or *a numerical representation of* $\succeq$ if

$$c' \succeq c \quad \iff \quad \mathcal{U}(c') \geq \mathcal{U}(c).$$

(a) Show that all $\mathcal{U}$ representing $\succeq$ must be *quasiconcave*, i.e., for all $c, c' \in \mathcal{C}$ and $\lambda \in [0, 1]$,

$$\mathcal{U}\left(\lambda c + (1 - \lambda)c'\right) \geq \min\{\mathcal{U}(c), \mathcal{U}(c')\}.$$

(b) Which axioms are needed for this result?

(c) Show by a counterexample that a preference order can be represented by a utility functional which is not concave.

Exercise 1.4 *This question is **optional**.*

(a) Show that any complete and transitive relation $\succeq$ induces an asymmetric and negative transitive order $\succ$ via

$$y \succ x \iff x \not\succeq y$$

Conversely, show that any asymmetric and negative transitive binary relation $\succ$ induces a complete and transitive binary relation $\succeq$.

In this question, we refer to an asymmetric and negative transitive relation $\succ$ as *preference order* and to the corresponding complete and transitive binary relation $\succeq$ as *weak preference order*. Moreover, we denote by $\mathcal{C}$ the set of consumption processes.

(b) Does every function $U : \mathcal{C} \to \mathbb{R}$ represent some preference order, i.e. an asymmetric and negative transitive relation?

Let $\succ$ be a preference relation on $\mathcal{C}$. A subset $\mathcal{Z}$ of $\mathcal{C}$ is called *order dense* if for any pair $x, y \in \mathcal{C}$ such that $x \succ y$ there exists some $z \in \mathcal{Z}$ with $x \succeq z \succeq y$.

(c) Show that, for the existence of a numerical representation of a preference relation $\succ$, it is necessary and sufficient that $\mathcal{C}$ contains a countable, order dense subset $\mathcal{Z}$.

(d) Find a preference order that does not admit a numerical representation. Which axioms from (P1)-(P5) does your example not satisfy?
Hint: Try the lexicographical order