Introduction to Mathematical Finance

Exercise sheet 2

Please hand in your solutions by Friday, 06/03/2020, 13:00 into Bálint Gersey's box next to HG G 53.2.

Exercise 2.1 Let $\mathcal{U}(c_0, c_T)$ be a numerical representation of a preference order $\succeq$ satisfying $(P1)-(P5)$ on $\mathcal{C}_+ := \mathbb{R}_+ \times \mathbb{R}^k_+$. Suppose $\mathcal{U}$ is concave, C^1 and satisfies the Inada condition,

$$\frac{\partial U}{\partial c_i}(c_0, c_1, \dots, c_k) \to \infty, \quad c_i \to 0;$$
$$\frac{\partial U}{\partial c_i}(c_0, c_1, \dots, c_k) \to 0, \quad c_i \to \infty;$$

where $(c_0, c_1, \dots, c_k) = (c_0, c_T(\omega_1), \dots, c_T(\omega_k))$.
An example of utility function satisfying the required condition is given by the square-root utility: $\mathcal{U}(c_0, c_T) = \sqrt{c_0} + \beta\mathbb{E}\left[\sqrt{c_T}\right]$ *for* $\beta > 0$.
Consider the optimization problem

$$\underset{c \in B_+(e,\pi)}{\text{maximize}} \quad \mathcal{U}(c_0, c_T) \tag{1}$$

where $B_+(e, \pi) := \{c \in \mathcal{C}_+ : \exists\theta \in \mathbb{R}^N \text{ with } c_0 \le e_0 - \theta \cdot \pi \text{ and } c_T \le e_T + \mathcal{D}\theta\}$ and $e \in \mathbb{R}_+ \times \mathbb{R}^k_+$ is a fixed endowment.
Recall from the lecture that the optimization problem (1) admits a solution if and only if the market is arbitrage-free. From now on, we suppose the No Arbitrage assumption.
The goal of this exercise is to show that the solution $\tilde{c} = (\tilde{c}_0, \tilde{c}_T) = (\tilde{c}_0, \tilde{c}_1, \dots, \tilde{c}_k)$ to (1) is determined by

$$\frac{\partial \mathcal{U}}{\partial c_0}(\tilde{c}_0, \tilde{c}_T)(-\pi^l) + \sum_{i=1}^{k} \frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T)\mathcal{D}^{l,i} = 0$$

for all $l = 1, \dots, N$, where $\frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T) > 0$ for all $i = 0, \dots, k$.

(a) Show that, under the assumption that $\frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T) > 0$ for all $i = 0, \dots, k$, the budget set is binding, i.e. there exists $\widetilde{\theta} \in \mathbb{R}^N$ such that the optimal consumption $\tilde{c}$ is generated by the endowment e and trading strategy $\widetilde{\theta}$.

Using that the budget set is binding, the optimization problem (1) is equivalent to finding a maximizer $\widetilde{\theta}$ of the function

$$f(\theta) := \mathcal{U}(e_0 - \theta \cdot \pi, e_T + \mathcal{D}\theta).$$

(b) Show that, in the interior of the optimization domain, a necessary and sufficient condition for the consumption generated by $\widetilde{\theta}$ to be optimal is

$$\frac{\partial \mathcal{U}}{\partial c_0}(\tilde{c}_0, \tilde{c}_T)(-\pi^l) + \sum_{i=1}^{k} \frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T)\mathcal{D}^{l,i} = 0$$

for all $l = 1, \ldots, N$, where $\frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T) > 0$ for all $i = 0, \ldots, k$.

Remains to show that the optimal solution $\tilde{c}$ is strictly positive, and hence, is in the interior of the optimization domain.

(c) Let $\varepsilon > 0$ satisfy $\tilde{c}_0 + \varepsilon\widetilde{\theta} \cdot \pi > 0$ and $\tilde{c}_j - \varepsilon(\mathcal{D}\widetilde{\theta})_j > 0$ for all $j = 1, \ldots, k$. Show that the consumption $c = (c_0, c_1, \ldots, c_T)$ defined by

$$\begin{aligned} c_0 &= \tilde{c}_0 + \varepsilon\widetilde{\theta} \cdot \pi \\ c_j &= \tilde{c}_j - \varepsilon(\mathcal{D}\widetilde{\theta})_j \quad j = 1, \ldots, k \end{aligned}$$

is in the budget set $B_+(e, \pi)$.
Hint: Consider the portfolio $(1 - \varepsilon)\widetilde{\theta}$

(d) Let c be the consumption defined in the previous question. Show that

$$\mathcal{U}(c) - \mathcal{U}(\tilde{c}) \geq \varepsilon \left(\widetilde{\theta} \cdot \pi \frac{\partial \mathcal{U}}{\partial c_o}(c) - \sum_{i=1}^{k} (\mathcal{D}\widetilde{\theta})_i \frac{\partial \mathcal{U}}{\partial c_i}(c) \right).$$

(e) Conclude, using Inada conditions, that the optimal consumption $\tilde{c}$ is strictly positive.

Solution 2.1

(a) We need to show that, although we optimize over all consumption in the budget set $B_+(e, \pi)$, the optimal consumption is attained at the boundary of $B_+(e, \pi)$, i.e. there exists $\widetilde{\theta} \in \mathbb{R}^N$ such that the optimal consumption $\tilde{c}$ is generated by the endowment e and trading strategy $\widetilde{\theta}$:

$$\begin{aligned} \tilde{c}_0 &= e_0 - \widetilde{\theta} \cdot \pi \\ \tilde{c}_j &= e_j + (\mathcal{D}\widetilde{\theta})_j \quad j = 1, \ldots, k \end{aligned}$$

Suppose by contradiction that either $\tilde{c}_0 < e_0 - \widetilde{\theta} \cdot \pi$ or $\tilde{c}_j < e_j + (\mathcal{D}\widetilde{\theta})_j$ for some $j = 1, \ldots, k$. The positivity assumption on the partial derivatives of $\mathcal{U}$ imply that the numerical representation is strictly increasing in all arguments and hence the utility corresponding to the consumption $(e_0 - \widetilde{\theta} \cdot \pi, e_1 + (\mathcal{D}\widetilde{\theta})_1, \ldots, e_k + (\mathcal{D}\widetilde{\theta})_k)$ would be strictly larger than $\mathcal{U}(\tilde{c}_0, \tilde{c}_T)$ contradicting the optimality of $\tilde{c}$.

(b) In the interior of the optimization domain, the first order optimality condition $\nabla f(\widetilde{\theta}) = 0$ is necessary and sufficient. A direct application of the chain rule gives the desired result.

(c) As suggested by the hint, consider the portfolio $(1-\varepsilon)\widetilde{\theta}$. The corresponding generated consumption is exactly c. Indeed,

$$\begin{aligned} c_0 &= e_0 - (1-\varepsilon)\widetilde{\theta}\cdot\pi = \tilde{c}_0 + \varepsilon\widetilde{\theta}\cdot\pi \\ c_j &= e_j + (1-\varepsilon)(\mathcal{D}\widetilde{\theta})_j = \tilde{c}_j - \varepsilon(\mathcal{D}\widetilde{\theta})_j \quad j = 1,\dots,k \end{aligned}$$

Moreover, $c \in \mathcal{C}_+$ by assumption, and therefore the consumption c indeed lies in the budget set $B_+(e,\pi)$.

(d) This is a direct consequence of the concavity of $\mathcal{U}$. Recall that every concave function lies, at any point, below the supporting hyperplane of its graph:

$$\mathcal{U}(\tilde{c}) \leq \mathcal{U}(c) + \nabla\mathcal{U}^T(c)(\tilde{c}-c) \tag{2}$$

Using that

$$\begin{aligned} \tilde{c}_0 - c_0 &= -\varepsilon\widetilde{\theta}\cdot\pi \\ \tilde{c}_j - c_j &= \varepsilon(\mathcal{D}\widetilde{\theta})_j \quad j = 1,\dots,k \end{aligned}$$

the concavity condition (2) reads

$$\mathcal{U}(c) - \mathcal{U}(\tilde{c}) \geq \varepsilon\left(\widetilde{\theta}\cdot\pi\frac{\partial\mathcal{U}}{\partial c_o}(c) - \sum_{i=1}^{k}(\mathcal{D}\widetilde{\theta})_i\frac{\partial\mathcal{U}}{\partial c_i}(c)\right).$$

(e) For a proof by contradiction, suppose that $\tilde{c}$ is not strictly positive, i.e. it admits at least one zero component. We have to distinguish two cases: $\tilde{c}_0 = 0$ or $\tilde{c}_j = 0$ for some $j = 1,\dots,k$.
Note that for ε small enough the lower bound derived in the previous question is strictly positive. Indeed, if $\tilde{c}_0 = 0$, the binding budget set condition $\tilde{c}_0 = e_0 - \widetilde{\theta}\cdot\pi$ implies $\widetilde{\theta}\cdot\pi = e_0 > 0$. Similarly, if $\tilde{c}_j = 0$ for some $j = 1,\dots,k$, the binding budget set condition $\tilde{c}_j = e_j + (\mathcal{D}\widetilde{\theta})_j$ implies $(\mathcal{D}\widetilde{\theta})_j = -e_j < 0$.
Moreover, as $\varepsilon \to 0$, we have $c \to \tilde{c}$, and hence Inada condition gives us

$$\lim_{\varepsilon\to 0}\frac{\partial\mathcal{U}}{\partial c_l}(c) = \infty$$

for all components $l = 0,1,\dots,k$ such that $\tilde{c}_l = 0$ and

$$\lim_{\varepsilon\to 0}\frac{\partial\mathcal{U}}{\partial c_m}(c) < \infty$$

for all components $m = 0,1,\dots,k$ such that $\tilde{c}_m \neq 0$.

Exercise 2.2 Consider a financial market as in the lecture. Show the following properties.

(a) For $\theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$, we have

$$\pi^T\theta = 0 \text{ and } \mathcal{D}\theta = 0 \quad \text{if and only if} \quad \theta \equiv 0 \in \mathbb{R}^L$$

From now on, we assume (NA) and suppose that for all consumption c in the budget set, the corresponding strategy θ satisfies $\theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$.

(b) Show that, under the above assumptions, the budget set $B(e, \pi)$ is closed in the Euclidean norm.

Remark: For a finite probability space $|\Omega| < \infty$, the result in b) holds even without the (NA) assumption.

Solution 2.2

(a) The sufficient condition is trivial. For the necessary one, let $\theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$. By definition, we have

$$\theta^T\widetilde{\theta} = 0 \text{ for all } \widetilde{\theta} \in \mathbb{R}^L \text{ such that } \pi^T\widetilde{\theta} = 0 = \mathcal{D}\widetilde{\theta}.$$

In particular, by our assumption, we can take $\widetilde{\theta} = \theta$ which gives $||\theta||^2 = 0$, and hence $\theta \equiv 0$.

(b) The budget set under our additional assumption is given by

$$B(e, \pi) := \{c \in \mathcal{C} = \mathbb{R}^{k+1} : \exists \theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp \text{ with } c \leq e + \bar{\mathcal{D}}\theta\}.$$

Take a sequence of consumption $(c_n)_{n\in\mathbb{N}}$ in the budget set that converge to a certain consumption c_∞. We need to show $c_\infty \in B(e, \pi)$. Using that $c_n \in B(e, \pi)$, we have for all n:

- $c_n \in \mathcal{C} = \mathbb{R}^{k+1}$
- $c_n \leq e + \bar{\mathcal{D}}\theta_n$ for some $\theta_n \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$

Suppose the sequence θ_n is bounded. Then it admits a converging sub-sequence $\theta_{n_j} \to \theta_\infty$, and taking the limit along the sub-sequence gives

- $c_\infty \in \mathcal{C} = \mathbb{R}^{k+1}$
- $c_\infty \leq e + \bar{\mathcal{D}}\theta_\infty$

where $\theta_\infty \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$ (convince yourself). This shows that, if the sequence $(\theta_n)_n$ is bounded, we have $c \in B(e, \pi)$ and hence $B(e, \pi)$ is sequentially closed in the metric space $\mathcal{C}$ and therefore is closed.
Remains to prove that the sequence $(\theta_n)_n$ is bounded. Suppose, for a proof

by contradiction, that $|\theta_n| \to \infty$ and consider the bounded sequence $\hat{\theta}_n := \frac{\theta_n}{|\theta_n|}$. Since the sequence $(\hat{\theta}_n)_n$ is bounded, we can extract a converging sub-sequence $\hat{\theta}_{n_j} \to \xi_\infty$ for some ξ_∞. It is easy to see that $|\xi_\infty| = 1$ (use that the inner product is a continuous map). Using that $c_n \in B(e, \pi)$, we have

$$\frac{c_n}{|\theta_n|} \leq \frac{e}{|\theta_n|} + \bar{\mathcal{D}}\hat{\theta}_n$$

which, taking the limit over the converging sub-sequence of $\hat{\theta}$, gives

$$0 \leq \bar{\mathcal{D}}\xi_\infty.$$

The (NA) assumption implies $\bar{\mathcal{D}}\xi_\infty = 0$. Note that $\theta_n \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$ implies that $\xi_\infty \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$, and hence we can use the result from a) to conclude $\xi_\infty = 0$.. This contradicts that $|\xi_\infty| = 1$ and hence the sequence $(\theta_n)_n$ is bounded.

Exercise 2.3 Consider the one-step *binomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u and d with $u > d$.

(a) Show that this market is free of arbitrage if and only if $u > r > d$.

(b) Construct an arbitrage opportunity for a market where $u = r > d$.

Solution 2.3

(a) The market being arbitrage-free is equivalent to the existence of a probability measure $Q = (q_u, q_d)$ with

$$E_Q\left[\frac{D^1}{D^0}\right] = q_u \frac{1+u}{1+r} + q_d \frac{1+d}{1+r} = 1,$$

$q_u + q_d = 1$, $q_u > 0$, and $q_d > 0$. These equalities are satisfied by

$$q_u = \frac{r-d}{u-d} \quad \text{and} \quad q_d = \frac{u-r}{u-d},$$

which satisfy the positivity conditions if and only if $r \in (d, u)$.

(b) If $u = r$, the risky asset can only lose value, relative to the risk-free asset. An arbitrage of the first kind is therefore given by ("go long risk-free asset and short risky asset")

$$\vartheta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This strategy costs $\vartheta \cdot \pi = 0$ at time 0 and yields

$$\mathcal{D}\vartheta = \begin{pmatrix} 0 \\ r-d \end{pmatrix}$$

at time T.

Exercise 2.4 Consider the one-step *trinomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u, m and d with $u > m > d$ and $u > r > d$.

(a) Show that $\mathbb{P}(D^0)$ is convex.

(b) Calculate the set $\mathbb{P}(D^0)$ of equivalent martingale measures.

Hint: Use the probability of the 'middle outcome' as a parameter in a parametrization of $\mathbb{P}(D^0)$ as a line segment in $\mathbb{R}^3$.

(c) Denote by $\mathbb{P}_a(D^0)$ the set of all martingale measures Q which are absolutely continuous with respect to P, i.e., $Q \ll P$. An element $R \in \mathbb{P}_a(D^0)$ is an *extreme point* if $R = \lambda Q + (1-\lambda)Q'$ with $0 < \lambda < 1$ and $Q, Q' \in \mathbb{P}_a(D^0)$ implies $Q = Q'$, i.e., R cannot be written as a strict convex combination of elements in $\mathbb{P}_a(D^0)$.

Find the extreme points of $\mathbb{P}_a(D^0)$ and represent $\mathbb{P}(D^0)$ by writing it as a (strict) convex combination of such extreme points. Verify that this coincides with the answer found above.

Solution 2.4

(a) Let Q^1 and Q^2 be two equivalent martingale measures and define $Q = \lambda Q^1 + (1-\lambda)Q^2$ for $\lambda \in [0, 1]$. Clearly $Q[\Omega] = 1$ and $Q[\{\omega_k\}] > 0$ for all k. Furthermore,

$$E_Q\left[\frac{D^\ell}{D^0}\right] = \lambda E_{Q^1}\left[\frac{D^\ell}{D^0}\right] + (1-\lambda)E_{Q^2}\left[\frac{D^\ell}{D^0}\right] = \lambda\pi^\ell + (1-\lambda)\pi^\ell = \pi^\ell,$$

for all ℓ, showing that $Q \in \mathbb{P}(D^0)$. Since λ was arbitrary, $\mathbb{P}(D^0)$ is convex.

(b) Let Q be any probability measure on $\mathcal{F}$ and $q_i = Q[\{\omega_i\}]$ for $i \in \{u, m, d\}$. Now write down the conditions on q_i:

$$\begin{aligned} \pi^1 &= E_Q\left[\frac{D^1}{D^0}\right], && \text{(Martingale property)} \\ &= \frac{(1+u)q_u + (1+m)q_m + (1+d)q_d}{1+r}\pi^1, \\ 1 &= q_u + q_m + q_d, && (Q[\Omega] = 1) \\ q_i &\in (0, 1), \quad i \in \{u, m, d\}. && (Q \approx P) \end{aligned}$$

As suggested in the hint, we parametrize this set by choosing $q_m = \lambda$. Using the two equations then yields

$$q_u = \frac{(r-d)-(m-d)\lambda}{u-d},$$
$$q_d = \frac{(u-r)-(u-m)\lambda}{u-d}.$$

Now we just have to restrict λ according to the third condition. This amounts to choosing λ such that

$$q_m \in (0,1) \Leftrightarrow \lambda \in (0,1),$$
$$q_u \in (0,1) \Leftrightarrow \lambda \in \left(\frac{r-u}{m-d}, \frac{r-d}{m-d}\right),$$
$$q_d \in (0,1) \Leftrightarrow \lambda \in \left(\frac{d-r}{u-m}, \frac{u-r}{u-m}\right).$$

Since $u > m > d$ and $u > r > d$ this reduces to

$$\lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right).$$

Hence, with the identification of $\mathbb{P}$ as a subset of $[0,1]^3$,

$$\mathbb{P} = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d}\right) : \lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right) \right\}.$$

(c) The extreme points can be found in two ways. First, one could calculate $\mathbb{P}_a(D^0)$ explicitly to obtain the closure of $\mathbb{P}$ found above and setting the parameter λ to its smallest and largest values. More precisely,

$$\mathbb{P}_a = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d}\right) : \lambda \in \left[0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right] \right\}.$$

Alternatively, since $\mathbb{P}_a$ is the intersection of the two planes ($Q[\Omega] = 1$) and (Martingale property) in the closed, positive orthant, the extreme points lie on the boundary of this orthant. Therefore, one could find solutions to (Martingale property) and ($Q[\Omega] = 1$) with $q_i \geq 0$ for all $i \in \{u, m, d\}$ and $q_i = 0$ for at least one $i \in \{u, m, d\}$. These points are given by

$$Q_1 = \left(\frac{r-d}{u-d}, 0, \frac{u-r}{u-d}\right), \quad \text{and} \quad Q_2 = \begin{cases} \left(0, \frac{r-d}{m-d}, \frac{m-r}{m-d}\right) & \text{if } m \geq r, \\ \left(\frac{r-m}{u-m}, \frac{u-r}{u-m}, 0\right) & \text{if } m < r. \end{cases}$$

Therefore,

$$\mathbb{P} = \{(1-\alpha)Q_1 + \alpha Q_2 : \alpha \in (0,1)\}.$$

We verify for $m \geq r$. The other case works analogously. For these parameters,

$$\lambda \in \left(0, \frac{r-d}{m-d}\right).$$

Let $\alpha = \lambda \frac{m-d}{r-d}$. Then $\alpha \in (0,1)$ and an element in $\mathbb{P}$ is given by

$$\begin{aligned}
(1-\alpha)Q_1 + \alpha Q_2 &= \left(\frac{(r-d)-(r-d)\alpha}{u-d}, \frac{r-d}{m-d}\alpha, \frac{(u-r)-(u-r)\alpha}{u-d} + \frac{m-r}{m-d}\alpha\right) \\
&= \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{u-r}{u-d} + \underbrace{\left(\frac{m-r}{m-d} - \frac{u-r}{u-d}\right)}_{-\frac{u-m}{u-d}\frac{r-d}{m-d}}\alpha\right) \\
&= \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{u-r}{u-d} - \frac{u-m}{u-d}\lambda\right),
\end{aligned}$$

which is of the same form as in the previous exercise.

Note: When $m \neq r$, Q_1 and Q_2 are both EMMs for the binomial markets obtained when one of the three points is removed.