Introduction to Mathematical Finance

Exercise sheet 2

Please hand in your solutions by Friday, 06/03/2020, 13:00 into Bálint Gersey's box next to HG G 53.2.

Exercise 2.1 Let $\mathcal{U}(c_0, c_T)$ be a numerical representation of a preference order $\succeq$ satisfying $(P1) - (P5)$ on $\mathcal{C}_+ := \mathbb{R}_+ \times \mathbb{R}_+^k$. Suppose $\mathcal{U}$ is concave, C^1 and satisfies the Inada condition,

$$\frac{\partial U}{\partial c_i}(c_0, c_1, \dots, c_k) \to \infty, \quad c_i \to 0;$$
$$\frac{\partial U}{\partial c_i}(c_0, c_1, \dots, c_k) \to 0, \quad c_i \to \infty;$$

where $(c_0, c_1, \dots, c_k) = (c_0, c_T(\omega_1), \dots, c_T(\omega_k))$.
An example of utility function satisfying the required condition is given by the square-root utility: $\mathcal{U}(c_0, c_T) = \sqrt{c_0} + \beta \mathbb{E}\left[\sqrt{c_T}\right]$ for $\beta > 0$.
Consider the optimization problem

$$\underset{c \in B_+(e,\pi)}{\text{maximize}} \quad \mathcal{U}(c_0, c_T) \tag{1}$$

where $B_+(e, \pi) := \{c \in \mathcal{C}_+ : \exists \theta \in \mathbb{R}^N \text{ with } c_0 \leq e_0 - \theta \cdot \pi \text{ and } c_T \leq e_T + \mathcal{D}\theta\}$ and $e \in \mathbb{R}_+ \times \mathbb{R}_+^k$ is a fixed endowment.
Recall from the lecture that the optimization problem (1) admits a solution if and only if the market is arbitrage-free. From now on, we suppose the No Arbitrage assumption. The goal of this exercise is to show that the solution $\tilde{c} = (\tilde{c}_0, \tilde{c}_T) = (\tilde{c}_0, \tilde{c}_1, \dots, \tilde{c}_k)$ to (1) is determined by

$$\frac{\partial \mathcal{U}}{\partial c_0}(\tilde{c}_0, \tilde{c}_T)(-\pi^l) + \sum_{i=1}^{k} \frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T)\mathcal{D}^{l,i} = 0$$

for all $l = 1, \dots, N$, where $\frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T) > 0$ for all $i = 0, \dots, k$.

(a) Show that, under the assumption that $\frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T) > 0$ for all $i = 0, \dots, k$, the budget set is binding, i.e. there exists $\widetilde{\theta} \in \mathbb{R}^N$ such that the optimal consumption $\tilde{c}$ is generated by the endowment e and trading strategy $\widetilde{\theta}$.

Using that the budget set is binding, the optimization problem (1) is equivalent to finding a maximizer $\widetilde{\theta}$ of the function

$$f(\theta) := \mathcal{U}(e_0 - \theta \cdot \pi, e_T + \mathcal{D}\theta).$$

(b) Show that, in the interior of the optimization domain, a necessary and sufficient condition for the consumption generated by $\widetilde{\theta}$ to be optimal is

$$\frac{\partial \mathcal{U}}{\partial c_0}(\tilde{c}_0, \tilde{c}_T)(-\pi^l) + \sum_{i=1}^{k} \frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T)\mathcal{D}^{l,i} = 0$$

for all $l = 1, \ldots, N$, where $\frac{\partial \mathcal{U}}{\partial c_i}(\tilde{c}_0, \tilde{c}_T) > 0$ for all $i = 0, \ldots, k$.

Remains to show that the optimal solution $\tilde{c}$ is strictly positive, and hence, is in the interior of the optimization domain.

(c) Let $\varepsilon > 0$ satisfy $\tilde{c}_0 + \varepsilon\widetilde{\theta} \cdot \pi > 0$ and $\tilde{c}_j - \varepsilon(\mathcal{D}\widetilde{\theta})_j > 0$ for all $j = 1, \ldots, k$. Show that the consumption $c = (c_0, c_1, \ldots, c_T)$ defined by

$$\begin{aligned} c_0 &= \tilde{c}_0 + \varepsilon\widetilde{\theta} \cdot \pi \\ c_j &= \tilde{c}_j - \varepsilon(\mathcal{D}\widetilde{\theta})_j \quad j = 1, \ldots, k \end{aligned}$$

is in the budget set $B_+(e, \pi)$.
Hint: Consider the portfolio $(1 - \varepsilon)\widetilde{\theta}$

(d) Let c be the consumption defined in the previous question. Show that

$$\mathcal{U}(c) - \mathcal{U}(\tilde{c}) \geq \varepsilon \left(\widetilde{\theta} \cdot \pi \frac{\partial \mathcal{U}}{\partial c_o}(c) - \sum_{i=1}^{k} (\mathcal{D}\widetilde{\theta})_i \frac{\partial \mathcal{U}}{\partial c_i}(c) \right).$$

(e) Conclude, using Inada conditions, that the optimal consumption $\tilde{c}$ is strictly positive.

Exercise 2.2 Consider a financial market as in the lecture. Show the following properties.

(a) For $\theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$, we have

$$\pi^T\theta = 0 \text{ and } \mathcal{D}\theta = 0 \quad \text{if and only if} \quad \theta \equiv 0 \in \mathbb{R}^L$$

From now on, we assume (NA) and suppose that for all consumption c in the budget set, the corresponding strategy θ satisfies $\theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$.

(b) Show that, under the above assumptions, the budget set $B(e, \pi)$ is closed in the Euclidean norm.

Remark: For a finite probability space $|\Omega| < \infty$, the result in b) holds even without the (NA) assumption.

Exercise 2.3 Consider the one-step *binomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1 + r & 1 + u \\ 1 + r & 1 + d \end{pmatrix},$$

for some $r > -1$, u and d with $u > d$.

(a) Show that this market is free of arbitrage if and only if $u > r > d$.

(b) Construct an arbitrage opportunity for a market where $u = r > d$.

Exercise 2.4 Consider the one-step *trinomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u, m and d with $u > m > d$ and $u > r > d$.

(a) Show that $\mathbb{P}(D^0)$ is convex.

(b) Calculate the set $\mathbb{P}(D^0)$ of equivalent martingale measures.

Hint: Use the probability of the 'middle outcome' as a parameter in a parametrization of $\mathbb{P}(D^0)$ as a line segment in $\mathbb{R}^3$.

(c) Denote by $\mathbb{P}_a(D^0)$ the set of all martingale measures Q which are absolutely continuous with respect to P, i.e., $Q \ll P$. An element $R \in \mathbb{P}_a(D^0)$ is an *extreme point* if $R = \lambda Q + (1-\lambda)Q'$ with $0 < \lambda < 1$ and $Q, Q' \in \mathbb{P}_a(D^0)$ implies $Q = Q'$, i.e., R cannot be written as a strict convex combination of elements in $\mathbb{P}_a(D^0)$.

Find the extreme points of $\mathbb{P}_a(D^0)$ and represent $\mathbb{P}(D^0)$ by writing it as a (strict) convex combination of such extreme points. Verify that this coincides with the answer found above.