# Introduction to Mathematical Finance

## Solution sheet 2

### Solution 2.1

(a) We need to show that, although we optimize over all consumption in the budget set $B_+(e,\pi)$, the optimal consumption is attained at the boundary of $B_+(e,\pi)$, i.e. there exists $\widetilde{\theta}\in\mathbb{R}^N$ such that the optimal consumption $\tilde{c}$ is generated by the endowment $e$ and trading strategy $\widetilde{\theta}$:

$$\begin{aligned}\tilde{c}_0 &= e_0 - \widetilde{\theta}\cdot\pi\\ \tilde{c}_j &= e_j + (\mathcal{D}\widetilde{\theta})_j \quad j=1,\dots,k\end{aligned}$$

Suppose by contradiction that either $\tilde{c}_0 < e_0 - \widetilde{\theta}\cdot\pi$ or $\tilde{c}_j < e_j + (\mathcal{D}\widetilde{\theta})_j$ for some $j = 1,\dots,k$. The positivity assumption on the partial derivatives of $\mathcal{U}$ imply that the numerical representation is strictly increasing in all arguments and hence the utility corresponding to the consumption $(e_0 - \widetilde{\theta}\cdot\pi, e_1 + (\mathcal{D}\widetilde{\theta})_1, \dots, e_k + (\mathcal{D}\widetilde{\theta})_k)$ would be strictly larger than $\mathcal{U}(\tilde{c}_0,\tilde{c}_T)$ contradicting the optimality of $\tilde{c}$.

(b) In the interior of the optimization domain, the first order optimality condition $\nabla f(\widetilde{\theta}) = 0$ is necessary and sufficient. A direct application of the chain rule gives the desired result.

(c) As suggested by the hint, consider the portfolio $(1-\varepsilon)\widetilde{\theta}$. The corresponding generated consumption is exactly $c$. Indeed,

$$\begin{aligned}c_0 &= e_0 - (1-\varepsilon)\widetilde{\theta}\cdot\pi = \tilde{c}_0 + \varepsilon\widetilde{\theta}\cdot\pi\\ c_j &= e_j + (1-\varepsilon)(\mathcal{D}\widetilde{\theta})_j = \tilde{c}_j - \varepsilon(\mathcal{D}\widetilde{\theta})_j \quad j=1,\dots,k\end{aligned}$$

Moreover, $c\in\mathcal{C}_+$ by assumption, and therefore the consumption $c$ indeed lies in the budget set $B_+(e,\pi)$.

(d) This is a direct consequence of the concavity of $\mathcal{U}$. Recall that every concave function lies, at any point, below the supporting hyperplane of its graph:

$$\mathcal{U}(\tilde{c}) \le \mathcal{U}(c) + \nabla\mathcal{U}^T(c)(\tilde{c}-c) \tag{1}$$

Using that

$$\begin{aligned}\tilde{c}_0 - c_0 &= -\varepsilon\widetilde{\theta}\cdot\pi\\ \tilde{c}_j - c_j &= \varepsilon(\mathcal{D}\widetilde{\theta})_j \quad j=1,\dots,k\end{aligned}$$

the concavity condition (1) reads

$$\mathcal{U}(c) - \mathcal{U}(\tilde{c}) \geq \varepsilon \left( \widetilde{\theta} \cdot \pi \frac{\partial \mathcal{U}}{\partial c_o}(c) - \sum_{i=1}^{k} (\mathcal{D}\widetilde{\theta})_i \frac{\partial \mathcal{U}}{\partial c_i}(c) \right).$$

(e) For a proof by contradiction, suppose that $\tilde{c}$ is not strictly positive, i.e. it admits at least one zero component. We have to distinguish two cases: $\tilde{c}_0 = 0$ or $\tilde{c}_j = 0$ for some $j = 1, \ldots, k$.
Note that for $\varepsilon$ small enough the lower bound derived in the previous question is strictly positive. Indeed, if $\tilde{c}_0 = 0$, the binding budget set condition $\tilde{c}_0 = e_0 - \widetilde{\theta} \cdot \pi$ implies $\widetilde{\theta} \cdot \pi = e_0 > 0$. Similarly, if $\tilde{c}_j = 0$ for some $j = 1, \ldots, k$, the binding budget set condition $\tilde{c}_j = e_j + (\mathcal{D}\widetilde{\theta})_j$ implies $(\mathcal{D}\widetilde{\theta})_j = -e_j < 0$.
Moreover, as $\varepsilon \to 0$, we have $c \to \tilde{c}$, and hence Inada condition gives us

$$\lim_{\varepsilon \to 0} \frac{\partial \mathcal{U}}{\partial c_l}(c) = \infty$$

for all components $l = 0, 1, \ldots, k$ such that $\tilde{c}_l = 0$ and

$$\lim_{\varepsilon \to 0} \frac{\partial \mathcal{U}}{\partial c_m}(c) < \infty$$

for all components $m = 0, 1, \ldots, k$ such that $\tilde{c}_m \neq 0$.

**Solution 2.2**

(a) The sufficient condition is trivial. For the necessary one, let $\theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$. By definition, we have

$$\theta^T \widetilde{\theta} = 0 \text{ for all } \widetilde{\theta} \in \mathbb{R}^L \text{ such that } \pi^T \widetilde{\theta} = 0 = \mathcal{D}\widetilde{\theta}.$$

In particular, by our assumption, we can take $\widetilde{\theta} = \theta$ which gives $||\theta||^2 = 0$, and hence $\theta \equiv 0$.

(b) The budget set under our additional assumption is given by

$$B(e, \pi) := \{c \in \mathcal{C} = \mathbb{R}^{k+1} : \exists \theta \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp \text{ with } c \leq e + \bar{\mathcal{D}}\theta\}.$$

Take a sequence of consumption $(c_n)_{n \in \mathbb{N}}$ in the budget set that converge to a certain consumption $c_\infty$. We need to show $c_\infty \in B(e, \pi)$. Using that $c_n \in B(e, \pi)$, we have for all $n$:

- $c_n \in \mathcal{C} = \mathbb{R}^{k+1}$
- $c_n \leq e + \bar{\mathcal{D}}\theta_n$ for some $\theta_n \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$

Suppose the sequence $\theta_n$ is bounded. Then it admits a converging sub-sequence $\theta_{n_j} \to \theta_\infty$, and taking the limit along the sub-sequence gives

- $c_\infty \in \mathcal{C} = \mathbb{R}^{k+1}$
- $c_\infty \leq e + \bar{\mathcal{D}}\theta_\infty$

where $\theta_\infty \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$ (convince yourself). This shows that, if the sequence $(\theta_n)_n$ is bounded, we have $c \in B(e,\pi)$ and hence $B(e,\pi)$ is sequentially closed in the metric space $\mathcal{C}$ and therefore is closed.
Remains to prove that the sequence $(\theta_n)_n$ is bounded. Suppose, for a proof by contradiction, that $|\theta_n| \to \infty$ and consider the bounded sequence $\hat{\theta}_n := \frac{\theta_n}{|\theta_n|}$. Since the sequence $(\hat{\theta}_n)_n$ is bounded, we can extract a converging sub-sequence $\hat{\theta}_{n_j} \to \xi_\infty$ for some $\xi_\infty$. It is easy to see that $|\xi_\infty| = 1$ (use that the inner product is a continuous map). Using that $c_n \in B(e,\pi)$, we have

$$\frac{c_n}{|\theta_n|} \leq \frac{e}{|\theta_n|} + \bar{\mathcal{D}}\hat{\theta}_n$$

which, taking the limit over the converging sub-sequence of $\hat{\theta}$, gives

$$0 \leq \bar{\mathcal{D}}\xi_\infty.$$

The (NA) assumption implies $\bar{\mathcal{D}}\xi_\infty = 0$. Note that $\theta_n \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$ implies that $\xi_\infty \in \ker(\pi^T)^\perp \cap \ker(\mathcal{D})^\perp$, and hence we can use the result from a) to conclude $\xi_\infty = 0$.. This contradicts that $|\xi_\infty| = 1$ and hence the sequence $(\theta_n)_n$ is bounded.

**Solution 2.3**

(a) The market being arbitrage-free is equivalent to the existence of a probability measure $Q = (q_u, q_d)$ with

$$E_Q\left[\frac{D^1}{D^0}\right] = q_u\frac{1+u}{1+r} + q_d\frac{1+d}{1+r} = 1,$$

$q_u + q_d = 1$, $q_u > 0$, and $q_d > 0$. These equalities are satisfied by

$$q_u = \frac{r-d}{u-d} \quad \text{and} \quad q_d = \frac{u-r}{u-d},$$

which satisfy the positivity conditions if and only if $r \in (d,u)$.

(b) If $u = r$, the risky asset can only lose value, relative to the risk-free asset. An arbitrage of the first kind is therefore given by ("go long risk-free asset and short risky asset")

$$\vartheta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This strategy costs $\vartheta \cdot \pi = 0$ at time 0 and yields

$$\mathcal{D}\vartheta = \begin{pmatrix} 0 \\ r-d \end{pmatrix}$$

at time $T$.

**Solution 2.4**

(a) Let $Q^1$ and $Q^2$ be two equivalent martingale measures and define $Q = \lambda Q^1 + (1-\lambda)Q^2$ for $\lambda \in [0,1]$. Clearly $Q[\Omega] = 1$ and $Q[\{\omega_k\}] > 0$ for all $k$. Furthermore,

$$E_Q\left[\frac{D^\ell}{D^0}\right] = \lambda E_{Q^1}\left[\frac{D^\ell}{D^0}\right] + (1-\lambda)E_{Q^2}\left[\frac{D^\ell}{D^0}\right] = \lambda\pi^\ell + (1-\lambda)\pi^\ell = \pi^\ell,$$

for all $\ell$, showing that $Q \in \mathbb{P}(D^0)$. Since $\lambda$ was arbitrary, $\mathbb{P}(D^0)$ is convex.

(b) Let $Q$ be any probability measure on $\mathcal{F}$ and $q_i = Q[\{\omega_i\}]$ for $i \in \{u,m,d\}$. Now write down the conditions on $q_i$:

$$\begin{aligned}
\pi^1 &= E_Q\left[\frac{D^1}{D^0}\right], \\
&= \frac{(1+u)q_u + (1+m)q_m + (1+d)q_d}{1+r}\pi^1, && \text{(Martingale property)}\\
1 &= q_u + q_m + q_d, && (Q[\Omega]=1)\\
q_i &\in (0,1), \quad i \in \{u,m,d\}. && (Q \approx P)
\end{aligned}$$

As suggested in the hint, we parametrize this set by choosing $q_m = \lambda$. Using the two equations then yields

$$\begin{aligned}
q_u &= \frac{(r-d)-(m-d)\lambda}{u-d},\\
q_d &= \frac{(u-r)-(u-m)\lambda}{u-d}.
\end{aligned}$$

Now we just have to restrict $\lambda$ according to the third condition. This amounts to choosing $\lambda$ such that

$$\begin{aligned}
q_m \in (0,1) &\Leftrightarrow \lambda \in (0,1),\\
q_u \in (0,1) &\Leftrightarrow \lambda \in \left(\frac{r-u}{m-d}, \frac{r-d}{m-d}\right),\\
q_d \in (0,1) &\Leftrightarrow \lambda \in \left(\frac{d-r}{u-m}, \frac{u-r}{u-m}\right).
\end{aligned}$$

Since $u > m > d$ and $u > r > d$ this reduces to

$$\lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right).$$

Hence, with the identification of $\mathbb{P}$ as a subset of $[0,1]^3$,

$$\mathbb{P} = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d}\right) : \lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right) \right\}.$$

(c) The extreme points can be found in two ways. First, one could calculate $\mathbb{P}_a(D^0)$ explicitly to obtain the closure of $\mathbb{P}$ found above and setting the parameter $\lambda$ to its smallest and largest values. More precisely,

$$\mathbb{P}_a = \left\{ \left( \frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d} \right) : \lambda \in \left[0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right] \right\}.$$

Alternatively, since $\mathbb{P}_a$ is the intersection of the two planes ($Q[\Omega]=1$) and (Martingale property) in the closed, positive orthant, the extreme points lie on the boundary of this orthant. Therefore, one could find solutions to (Martingale property) and ($Q[\Omega]=1$) with $q_i \geq 0$ for all $i \in \{u,m,d\}$ and $q_i = 0$ for at least one $i \in \{u,m,d\}$. These points are given by

$$Q_1 = \left(\frac{r-d}{u-d}, 0, \frac{u-r}{u-d}\right), \quad \text{and} \quad Q_2 = \begin{cases} \left(0, \frac{r-d}{m-d}, \frac{m-r}{m-d}\right) & \text{if } m \geq r, \\ \left(\frac{r-m}{u-m}, \frac{u-r}{u-m}, 0\right) & \text{if } m < r. \end{cases}$$

Therefore,

$$\mathbb{P} = \{(1-\alpha)Q_1 + \alpha Q_2 : \alpha \in (0,1)\}.$$

We verify for $m \geq r$. The other case works analogously. For these parameters,

$$\lambda \in \left(0, \frac{r-d}{m-d}\right).$$

Let $\alpha = \lambda \frac{m-d}{r-d}$. Then $\alpha \in (0,1)$ and an element in $\mathbb{P}$ is given by

$$\begin{aligned}
(1-\alpha)Q_1 + \alpha Q_2 &= \left( \frac{(r-d)-(r-d)\alpha}{u-d}, \frac{r-d}{m-d}\alpha, \frac{(u-r)-(u-r)\alpha}{u-d} + \frac{m-r}{m-d}\alpha \right) \\
&= \left( \frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{u-r}{u-d} + \underbrace{\left(\frac{m-r}{m-d} - \frac{u-r}{u-d}\right)}_{-\frac{u-m}{u-d}\frac{r-d}{m-d}} \alpha \right) \\
&= \left( \frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{u-r}{u-d} - \frac{u-m}{u-d}\lambda \right),
\end{aligned}$$

which is of the same form as in the previous exercise.

*Note:* When $m \neq r$, $Q_1$ and $Q_2$ are both EMMs for the binomial markets obtained when one of the three points is removed.