Introduction to Mathematical Finance

Solution sheet 2

Solution 2.1

(a) We need to show that, although we optimize over all consumption in the budget set $B_+(e, \pi)$, the optimal consumption is attained at the boundary of $B_+(e, \pi)$, i.e. there exists $\tilde{\theta} \in \mathbb{R}^N$ such that the optimal consumption $\tilde{c}$ is generated by the endowment $e$ and trading strategy $\tilde{\theta}$:

$$\tilde{c}_0 = e_0 - \tilde{\theta} \cdot \pi$$
$$\tilde{c}_j = e_j + (D\tilde{\theta})_j \quad j = 1, \ldots, k$$

Suppose by contradiction that either $\tilde{c}_0 < e_0 - \tilde{\theta} \cdot \pi$ or $\tilde{c}_j < e_j + (D\tilde{\theta})_j$ for some $j = 1, \ldots, k$. The positivity assumption on the partial derivatives of $U$ imply that the numerical representation is strictly increasing in all arguments and hence the utility corresponding to the consumption $(e_0 - \tilde{\theta} \cdot \pi, e_1 + (D\tilde{\theta})_1, \ldots, e_k + (D\tilde{\theta})_k)$ would be strictly larger than $U(\tilde{c}_0, \tilde{c}_T)$ contradicting the optimality of $\tilde{c}$.

(b) In the interior of the optimization domain, the first order optimality condition $\nabla f(\tilde{\theta}) = 0$ is necessary and sufficient. A direct application of the chain rule gives the desired result.

(c) As suggested by the hint, consider the portfolio $(1 - \varepsilon)\tilde{\theta}$. The corresponding generated consumption is exactly $c$. Indeed,

$$c_0 = e_0 - (1 - \varepsilon)\tilde{\theta} \cdot \pi = \tilde{c}_0 + \varepsilon\tilde{\theta} \cdot \pi$$
$$c_j = e_j + (1 - \varepsilon)(D\tilde{\theta})_j = \tilde{c}_j - \varepsilon(D\tilde{\theta})_j \quad j = 1, \ldots, k$$

Moreover, $c \in C_+$ by assumption, and therefore the consumption $c$ indeed lies in the budget set $B_+(e, \pi)$.

(d) This is a direct consequence of the concavity of $U$. Recall that every concave function lies, at any point, below the supporting hyperplane of its graph:

$$U(\tilde{c}) \leq U(c) + \nabla U(c)^T(\tilde{c} - c) \quad (1)$$

Using that

$$\tilde{c}_0 - c_0 = -\varepsilon\tilde{\theta} \cdot \pi$$
$$\tilde{c}_j - c_j = \varepsilon(D\tilde{\theta})_j \quad j = 1, \ldots, k$$
the concavity condition reads
\[ U(c) - U(\tilde{c}) \geq \varepsilon \left( \tilde{\theta} \cdot \pi \frac{\partial U}{\partial c_o}(c) - \sum_{i=1}^{k} (D\tilde{\theta})_i \frac{\partial U}{\partial c_i}(c) \right). \]

(e) For a proof by contradiction, suppose that \( \tilde{c} \) is not strictly positive, i.e. it admits at least one zero component. We have to distinguish two cases: \( \tilde{c}_0 = 0 \) or \( \tilde{c}_j = 0 \) for some \( j = 1, \ldots, k \).

Note that for \( \varepsilon \) small enough the lower bound derived in the previous question is strictly positive. Indeed, if \( \tilde{c}_0 = 0 \), the binding budget set condition \( \tilde{c}_0 = e_0 - \tilde{\theta} \cdot \pi \) implies \( \tilde{\theta} \cdot \pi = e_0 > 0 \). Similarly, if \( \tilde{c}_j = 0 \) for some \( j = 1, \ldots, k \), the binding budget set condition \( \tilde{c}_j = e_j + (D\tilde{\theta})_j \) implies \( (D\tilde{\theta})_j = -e_j < 0 \).

Moreover, as \( \varepsilon \to 0 \), we have \( c \to \tilde{c} \), and hence Inada condition gives us
\[ \lim_{\varepsilon \to 0} \frac{\partial U}{\partial c_l}(c) = \infty \]
for all components \( l = 0, 1, \ldots, k \) such that \( \tilde{c}_l = 0 \) and
\[ \lim_{\varepsilon \to 0} \frac{\partial U}{\partial c_m}(c) < \infty \]
for all components \( m = 0, 1, \ldots, k \) such that \( \tilde{c}_m \neq 0 \).

Solution 2.2

(a) The sufficient condition is trivial. For the necessary one, let \( \theta \in \ker(\pi^T)^\perp \cap \ker(D)^\perp \). By definition, we have
\[ \theta^T\bar{\theta} = 0 \]
for all \( \bar{\theta} \in \mathbb{R}^L \) such that \( \pi^T\bar{\theta} = 0 = D\bar{\theta} \).

In particular, by our assumption, we can take \( \bar{\theta} = \theta \) which gives \( \|	heta\|^2 = 0 \), and hence \( \theta \equiv 0 \).

(b) The budget set under our additional assumption is given by
\[ B(e, \pi) := \{ c \in \mathcal{C} = \mathbb{R}^{k+1} : \exists \theta \in \ker(\pi^T)^\perp \cap \ker(D)^\perp \text{ with } c \leq e + D\theta \}. \]

Take a sequence of consumption \( (c_n)_{n \in \mathbb{N}} \) in the budget set that converge to a certain consumption \( c_\infty \). We need to show \( c_\infty \in B(e, \pi) \). Using that \( c_n \in B(e, \pi) \), we have for all \( n \):
- \( c_n \in \mathcal{C} = \mathbb{R}^{k+1} \)
- \( c_n \leq e + D\theta_n \) for some \( \theta_n \in \ker(\pi^T)^\perp \cap \ker(D)^\perp \)

Suppose the sequence \( \theta_n \) is bounded. Then it admits a converging sub-sequence \( \theta_{n_j} \to \theta_\infty \), and taking the limit along the sub-sequence gives
• \( c_\infty \in \mathcal{C} = \mathbb{R}^{k+1} \)
• \( c_\infty \leq e + \overline{D} \theta_\infty \)

where \( \theta_\infty \in \ker(\pi^T)^\perp \cap \ker(D)^\perp \) (convince yourself). This shows that, if the sequence \((\theta_n)_n\) is bounded, we have \( c \in B(e, \pi) \) and hence \( B(e, \pi) \) is sequentially closed in the metric space \( \mathcal{C} \) and therefore is closed.

Remains to prove that the sequence \((\theta_n)_n\) is bounded. Suppose, for a proof by contradiction, that \(|\theta_n| \to \infty\) and consider the bounded sequence \( \hat{\theta}_n := \frac{\theta_n}{|\theta_n|} \).

Since the sequence \((\hat{\theta}_n)_n\) is bounded, we can extract a converging sub-sequence \( \hat{\theta}_{n_j} \to \xi_\infty \) for some \( \xi_\infty \). It is easy to see that \(|\xi_\infty| = 1\) (use that the inner product is a continuous map). Using that \( c_n \in B(e, \pi) \), we have

\[
\frac{c_n}{|\theta_n|} \leq \frac{e}{|\theta_n|} + \overline{D} \hat{\theta}_n
\]

which, taking the limit over the converging sub-sequence of \( \hat{\theta} \), gives

\[
0 \leq \overline{D} \xi_\infty.
\]

The (NA) assumption implies \( \overline{D} \xi_\infty = 0 \). Note that \( \theta_n \in \ker(\pi^T)^\perp \cap \ker(D)^\perp \) implies that \( \xi_\infty \in \ker(\pi^T)^\perp \cap \ker(D)^\perp \), and hence we can use the result from a) to conclude \( \xi_\infty = 0 \). This contradicts that \(|\xi_\infty| = 1\) and hence the sequence \((\theta_n)_n\) is bounded.

**Solution 2.3**

(a) The market being arbitrage-free is equivalent to the existence of a probability measure \( Q = (q_u, q_d) \) with

\[
E_Q \left[ \frac{D^1}{D^0} \right] = q_u \frac{1 + u}{1 + r} + q_d \frac{1 + d}{1 + r} = 1,
\]

\( q_u + q_d = 1 \), \( q_u > 0 \), and \( q_d > 0 \). These equalities are satisfied by

\[
q_u = \frac{r - d}{u - d} \quad \text{and} \quad q_d = \frac{u - r}{u - d},
\]

which satisfy the positivity conditions if and only if \( r \in (d, u) \).

(b) If \( u = r \), the risky asset can only lose value, relative to the risk-free asset. An arbitrage of the first kind is therefore given by (“go long risk-free asset and short risky asset”)

\[
\vartheta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

This strategy costs \( \vartheta \cdot \pi = 0 \) at time 0 and yields

\[
D \vartheta = \begin{pmatrix} 0 \\ r - d \end{pmatrix}
\]

at time \( T \).
Solution 2.4

(a) Let $Q^1$ and $Q^2$ be two equivalent martingale measures and define

$$Q = \lambda Q^1 + (1 - \lambda)Q^2$$

for $\lambda \in [0, 1]$. Clearly $Q[\Omega] = 1$ and $Q[\{\omega_k\}] > 0$ for all $k$. Furthermore,

$$E_Q \left[ \frac{D^\ell}{D^0} \right] = \lambda E_{Q^1} \left[ \frac{D^\ell}{D^0} \right] + (1 - \lambda)E_{Q^2} \left[ \frac{D^\ell}{D^0} \right] = \lambda \pi^\ell + (1 - \lambda)\pi^\ell = \pi^\ell,$$

for all $\ell$, showing that $Q \in \mathbb{P}(D^0)$. Since $\lambda$ was arbitrary, $\mathbb{P}(D^0)$ is convex.

(b) Let $Q$ be any probability measure on $\mathcal{F}$ and $q_i = Q[\{\omega_i\}]$ for $i \in \{u, m, d\}$.

Now write down the conditions on $q_i$:

$$\pi^1 = E_Q \left[ \frac{D^1}{D^0} \right],$$

(Martingale property)

$$= \frac{(1 + u)q_u + (1 + m)q_m + (1 + d)q_d}{1 + r},$$

$$1 = q_u + q_m + q_d,$$

$$q_i \in (0, 1), \quad i \in \{u, m, d\}.$$  

(Q[\Omega] = 1)

(Q \approx P)

As suggested in the hint, we parametrize this set by choosing $q_m = \lambda$. Using the two equations then yields

$$q_u = \frac{(r - d) - (m - d)\lambda}{u - d},$$

$$q_d = \frac{(u - r) - (u - m)\lambda}{u - d}.$$  

Now we just have to restrict $\lambda$ according to the third condition. This amounts to choosing $\lambda$ such that

$$q_m \in (0, 1) \Leftrightarrow \lambda \in (0, 1),$$

$$q_u \in (0, 1) \Leftrightarrow \lambda \in \left( \frac{r - u}{m - d}, \frac{r - d}{m - d} \right),$$

$$q_d \in (0, 1) \Leftrightarrow \lambda \in \left( \frac{d - r}{u - m}, \frac{u - r}{u - m} \right).$$

Since $u > m > d$ and $u > r > d$ this reduces to

$$\lambda \in \left( 0, \min \left\{ \frac{r - d}{m - d}, \frac{u - r}{u - m} \right\} \right).$$

Hence, with the identification of $\mathbb{P}$ as a subset of $[0, 1]^3$,

$$\mathbb{P} = \left\{ \left( \frac{(r - d) - (m - d)\lambda}{u - d}, \frac{(u - r) - (u - m)\lambda}{u - d} \right), \lambda \in \left( 0, \min \left\{ \frac{r - d}{m - d}, \frac{u - r}{u - m} \right\} \right) \right\}.$$
(c) The extreme points can be found in two ways. First, one could calculate \( P_a(D^0) \) explicitly to obtain the closure of \( P \) found above and setting the parameter \( \lambda \) to its smallest and largest values. More precisely,

\[
P_a = \left\{ \left( \frac{(r - d) - (m - d)\lambda}{u - d}, \lambda, \frac{(u - r) - (u - m)\lambda}{u - d} \right) : \lambda \in \left[0, \min \left\{ \frac{r - d}{m - d}, \frac{u - r}{u - m}\right\}\right]\right\}.
\]

Alternatively, since \( P_a \) is the intersection of the two planes \( \{Q[\Omega] = 1\} \) and \( \text{[Martingale property]} \) in the closed, positive orthant, the extreme points lie on the boundary of this orthant. Therefore, one could find solutions to \( \text{[Martingale property]} \) and \( \{Q[\Omega] = 1\} \) with \( q_i \geq 0 \) for all \( i \in \{u, m, d\} \) and \( q_i = 0 \) for at least one \( i \in \{u, m, d\} \). These points are given by

\[
Q_1 = \left( \frac{r - d}{u - d}, 0, \frac{u - r}{u - d} \right), \quad \text{and} \quad Q_2 = \begin{cases} 
\left( 0, \frac{r - d}{m - d}, \frac{m - r}{m - d} \right) & \text{if } m \geq r, \\
\left( \frac{r - m}{u - m}, \frac{u - r}{u - m}, 0 \right) & \text{if } m < r.
\end{cases}
\]

Therefore,

\[
P = \{ (1 - \alpha)Q_1 + \alpha Q_2 : \alpha \in (0, 1) \}.
\]

We verify for \( m \geq r \). The other case works analogously. For these parameters,

\[
\lambda \in \left(0, \frac{r - d}{m - d}\right).
\]

Let \( \alpha = \lambda \frac{m - d}{r - d} \). Then \( \alpha \in (0, 1) \) and an element in \( P \) is given by

\[
(1 - \alpha)Q_1 + \alpha Q_2 = \left( \frac{(r - d) - (r - d)\alpha}{u - d}, \frac{r - d}{m - d}\alpha, \frac{(u - r) - (u - r)\alpha}{u - d} + \frac{m - r}{m - d}\alpha \right)
\]

\[
= \left( \frac{(r - d) - (m - d)\lambda}{u - d}, \lambda, \frac{u - r}{u - d} + \left( \frac{m - r}{m - d} - \frac{u - r}{u - d} \right)\alpha \right)
\]

\[
= \left( \frac{(r - d) - (m - d)\lambda}{u - d}, \lambda, \frac{u - r}{u - d} - \frac{u - m}{u - d}\lambda \right),
\]

which is of the same form as in the previous exercise.

\textit{Note:} When \( m \neq r \), \( Q_1 \) and \( Q_2 \) are both EMMs for the binomial markets obtained when one of the three points is removed.