Introduction to Mathematical Finance

Exercise sheet 3

Please hand in your solutions until Friday, 13.03.2020, 13:00 into Bálint Gersey's box next to HG G 53.2.

Exercise 3.1 Let H be a payoff at time T and Ψ a consistent price system.

(a) Show that if H is attainable, then $R^H \in \mathrm{Span}(R^{D^\ell}, 0 \leq \ell \leq N)$.

(b) Suppose that Q is the EMM associated to Ψ and H is attainable. Also assume that D^0 is a bond with interest rate r. Compute $E_Q[R^H]$.

Solution 3.1

(a) Since H is attainable, there exists some $\vartheta \in \mathbb{R}^N$ such that $H = \mathcal{D}\vartheta$. We thus have

$$R^H = \frac{H}{\Psi(0,H)} - 1 = \frac{\mathcal{D}\vartheta - \vartheta \cdot \pi}{\vartheta \cdot \pi} = \sum_{\ell=0}^{N} \frac{\vartheta^\ell (D^\ell - \pi^\ell)}{\pi^\ell} \frac{\pi^\ell}{\vartheta \cdot \pi} = \sum_{\ell=0}^{N} \frac{\vartheta^\ell \pi^\ell}{\vartheta \cdot \pi} R^{D^\ell}.$$

where in the first equality we have used the definition of the return of H.

(b) By Theorem I.5.2, we know that there exists a bijection between set $\mathbb{P}$ of all EMMs Q and the set of all consistent price systems Ψ on $\mathcal{C}$ given by $E_Q[H] = \Psi(0, D^0 H) = (1+r)\Psi(0,H)$ (note that in the last equality we have used the linearity of the price systems Ψ and the fact that D^0 is a bond with interest rate r). Moreover since $R^H = \frac{H - \Psi(0,H)}{\Psi(0,H)}$, we have

$$E_Q[R^H] = \frac{E_Q[H] - \Psi(0,H)}{\Psi(0,H)} = r.$$

Exercise 3.2 Let Ψ be a consistent price system and H a payoff at time T. Suppose that the market is arbitrage-free and that H is attainable. Also assume that D^0 is a bond with interest rate r. Show that under any probability measure $Q \sim P$,

$$E_Q[R^H] - r = -\mathrm{Cov}_Q\left(\frac{dP^*}{dQ}, R^H\right),$$

where $P^* \in \mathbb{P}$ is the EMM associated to Ψ and Cov_Q denotes the covariance with respect to Q. *Hint: use the result from Exercise 3.1 (b)*

Solution 3.2 Set $Z^* := \frac{dP^*}{dQ}$. We compute, using $E_Q[Z^*] = 1$ and $E_{P^*}[R^H] = r$ from Exercise 3.1(b),

$$\begin{aligned}\mathrm{Cov}_Q(Z^*, R^H) &= E_Q[Z^* R^H] - E_Q[Z^R]E_Q[R^H] \\ &= E_{P^*}[R^H] - E_Q[R^H] \\ &= r - E_Q[R^H] \qquad .\end{aligned}$$

This completes the proof.

Note: Proving CAPM-type relation is much simpler if H is attainable.

Exercise 3.3 Recall the setup in Exercise 2.4, where

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix}$$

for some $r > -1$, u, m and d with $u \geq m \geq d$ and $u > r > d$. Denote by $\mathbb{P}_a$ the set of all martingale measures Q which are absolutely continuous with respect to P, i.e., $Q \ll P$.

(a) Show that $\mathbb{P}_a = \overline{\mathbb{P}}$. Here we identity $\mathbb{P}$ with a subset of $\mathbb{R}^K_+ = \mathbb{R}^3_+$ and denote by $^{-}$ the closure in $\mathbb{R}^K$.

(b) Use (a) to show that for any random variable X,

$$\sup_{Q \in \mathbb{P}} E_Q[X] = \sup_{Q \in \mathbb{P}_a} E_Q[X].$$

(c) Show that for any payoff H, the supremum

$$\sup_{Q \in \mathbb{P}_a} E_Q\left[\frac{H}{D^0}\right]$$

is attained in some $Q \in \mathbb{P}_a$. Does this imply that the market is complete?

Solution 3.3

(a) Recall from the solution of Exercise 2.4 that

$$\mathbb{P}_a = \left\{ \left(\frac{(r-d) - (m-d)\lambda}{u-d}, \lambda, \frac{(u-r) - (u-m)\lambda}{u-d} \right) : \lambda \in \left[0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right] \right\},$$

which is equal to $\overline{\mathbb{P}}$.

Alternative solution to (a). Because $u > r > d$, $\mathbb{P} \neq \varnothing$. Take $P^* \in \mathbb{P}$ and any $Q \in \mathbb{P}_a$. Consider $Q_\varepsilon := \varepsilon P^* + (1-\varepsilon)Q$ and observe that

- Q_ε is a martingale measure since both P^* and Q are martingale measures
- Q_ε and P are equivalent since $P^* \sim P$ and $Q << P$

Taking the limit as $\epsilon \to 0$, we get that $Q \in \overline{\mathbb{P}}$.

(b) Let R be an element in $\mathbb{P}_a$. Let Q be an arbitrary element in $\mathbb{P}$ and construct $Q^\varepsilon = \varepsilon Q + (1-\varepsilon)R$. Then $Q^\varepsilon \in \mathbb{P}$ for all $\varepsilon \in (0,1]$ by construction, and

$$\lim_{\varepsilon \searrow 0} E_{Q^\varepsilon}[X] = \lim_{\varepsilon \searrow 0} \varepsilon E_Q[X] + (1-\varepsilon)E_R[X] = E_R[X],$$

implying that

$$\sup_{Q \in \mathbb{P}} E_Q[X] \geq \sup_{Q \in \mathbb{P}_a} E_Q[X].$$

The converse inequality inequality is trivial.

(c) By (a), $\mathbb{P}_a$ is closed and bounded, so $\mathbb{P}_a$ is compact. This uses again that we identify $\mathbb{P}$ and $\mathbb{P}_a$ with subset of $\mathbb{R}_+^K$. Also the mapping

$$\begin{aligned} \ell : \mathbb{P}_a &\to \mathbb{R}, \\ Q &\mapsto E_Q[X] \end{aligned}$$

is linear and hence continuous because $\mathbb{P}_a$ is finite-dimensional here. Thus the supremum is in fact attained.

Clearly the above result does not depend on the particular values of r, u, m, d. All we need is $u > r > d$ to have $\mathbb{P} \neq \varnothing$. So we can adjust $\mathcal{D}$ to obtain an incomplete market by taking $u > m > d$, while the conclusion of (c) is still true.

Exercise 3.4 Let

$$\pi = \begin{pmatrix} 1 \\ 1000 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1.1 & 1200 \\ 1.1 & 1100 \\ 1.1 & 800 \end{pmatrix}.$$

This is similar to the example with the gold market from the lecture, but now with three possible outcomes. Denote by H the payoff of a put option with strike $K = 900$, i.e.,

$$H = (900 - D^1)^+ = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}.$$

(a) Find

$$\sup_{Q \in \mathbb{P}(D^0)} E_Q\left[\frac{H}{D^0}\right].$$

(b) Find

$$\inf\{\vartheta \cdot \pi : \mathcal{D}\vartheta \geq H\}.$$

(c) Construct a market with $\mathbb{P}_a \neq \overline{\mathbb{P}}$, where we use the notation from Exercise 3.3.

Solution 3.4 Note that $\mathcal{D}$ is of the form

$$\mathcal{D} = \begin{pmatrix} \pi^0(1+r) & \pi^1(1+u) \\ \pi^0(1+r) & \pi^1(1+m) \\ \pi^0(1+r) & \pi^1(1+d) \end{pmatrix},$$

with $r = 0.1$, $u = 0.2$, $m = 0.1$ and $d = -0.2$.

(a) From Exercise 2.4 we know that

$$\mathbb{P} = \left\{ \left(\frac{(r-d) - (m-d)\lambda}{u-d}, \lambda, \frac{(u-r) - (u-m)\lambda}{u-d} \right) : \lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right) \right\}.$$

Set $q_i = Q[\{\omega_i\}]$ for $i \in \{1,2,3\}$. Since $E_Q[H/D^0] = \sum_{i=1}^3 q_i H(\omega_i)/D^0 = \frac{100 q_3}{1.1}$, which is decreasing in λ, we find the supremum by setting $\lambda = 0$ to obtain

$$\sup_{Q \in \mathbb{P}} E_Q\left[\frac{H}{D^0}\right] = \frac{100}{1.1}\frac{1}{4} = \frac{25}{1.1} = 22.73.$$

(b) Writing down the condition $\mathcal{D}\vartheta \geq H$, we obtain the optimisation problem

$$\begin{aligned} \min \quad & \vartheta^0 + \pi^1\vartheta^1 \\ \text{s.t.} \quad & 11\vartheta^0 + 12\pi^1\vartheta^1 \geq 0, \\ & \vartheta^0 + \pi^1\vartheta^1 \geq 0, \\ & 11\vartheta^0 + 8\pi^1\vartheta^1 \geq 10H_3 = 1000. \end{aligned}$$

Note that the solution is found at an extreme point of the set of feasible solutions, and that the second condition cannot be satisfied with equality without violating the other two. Therefore, solving the outer two inequalities with equality gives

$$\vartheta^s = \begin{pmatrix} \frac{30}{11} \\ -\frac{5}{2\pi^1} \end{pmatrix} H_3$$

as a solution to the optimisation problem, and

$$\pi \cdot \vartheta^s = \frac{25}{1.1}.$$

(c) Extend the market with the asset H at the price $\pi_s(H)$. Denote by $\tilde{\mathbb{P}}$ (resp. $\tilde{\mathbb{P}}_a$) the set of all equivalent (resp. all absolutely continuous) martingale measures with the numéraire D^0 in the extended market. Then, from the characterisation of martingale measures to the original market, we conclude that

$$\tilde{\mathbb{P}}_a = \left\{ \left(\frac{3}{4}, 0, \frac{1}{4} \right) \right\},$$

(this is the measure corresponding to $\lambda = 0$) which is of course not an *equivalent* martingale measure. Hence, $\tilde{\mathbb{P}} = \varnothing$ and $\tilde{\mathbb{P}}_a \neq \overline{\tilde{\mathbb{P}}}$.

A simpler example is given by the market

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

where the only measure satisfying the martingale property is identified by $(0, 1)$, which is not equivalent to P.

Exercise 3.5 CAPM Analysis: Calculating stock Beta as a Regression. Recall that CAPM is a model that prices securities by examining the relationship between expected returns and risk. More precisely the model states that the return of a risky asset is given by

$$E[R_i] = r + \beta_i(E[R_m] - r) \tag{1}$$

where $E[R_i]$ and $E[R_m]$ are the expected returns of asset i and of the market respectively and r is the risk-free rate of interest. We have seen in the lecture that β_i has a closed from solution given by

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$$

The Beta of an asset is thus a measure of the sensitivity of its returns relative to a market benchmark (usually a market index). We could in principle calculate the Beta of a given asset using the above formula, however in practice it is easier to think of the Beta as the slope of the linear regression (1) and estimate it using Ordinary Least Squares method.

In this exercise we will estimate the Beta of Facebook relative to the S&P500 market index using historical data from 02/11/2014 to 19/11/2017. The data can be downloaded from the course web page ('FB.csv' and 'SP500.csv'). Your task is to complete the below Python code in order to perform an Ordinary Least Squares Regression with Statsmodel. If your code works well, you should get a Beta value close to 0.58 which is the Beta value of Facebook quoted on Yahoo Finance on the 19/11/2017.

```
# import libraries
import pandas as pd
import statsmodels.api as sm

'''
Download monthly prices of Facebook and S&P 500 index from
   2014 to 2017
CSV file downloaded from Yahoo File
start period: 02/11/2014
end period: 30/11/2017
period format: DD/MM/YEAR
'''

# Step 1: Use pandas read_csv method to load the two csv
   files downloaded from the course web page

fb = # TO DO #
sp_500 = # TO DO #

# joining the closing prices of the two datasets
```

```python
monthly_prices = pd.concat([fb['Close'], sp_500['Close']],
    axis=1)
monthly_prices.columns = ['FB', 'SP500']

# check the head of the dataframe
print(monthly_prices.head())

# calculate monthly returns
monthly_returns = # TO DO #
clean_monthly_returns = monthly_returns.dropna(axis=0)

# split dependent and independent variable
X = # TO DO #
y = # TO DO #

# Add a constant to the independent value
X1 = sm.add_constant(X)

# make regression model
model = # TO DO #

# fit model and print results
results = # TO DO #
print(results.summary())
```

Make sure you understand the summary of the linear regression and that in particular you can find the corresponding Beta value.

Solution 3.5

```python

# import libraries
import pandas as pd
import statsmodels.api as sm

'''
Download monthly prices of Facebook and S&P 500 index from
    2014 to 2017
CSV file downloaded from Yahoo File
start period: 02/11/2014
end period: 30/11/2017
period format: DD/MM/YEAR
'''
fb = pd.read_csv('FB.csv', parse_dates=True, index_col='Date
    ',)
sp_500 = pd.read_csv('SP500.csv', parse_dates=True,
    index_col='Date')
```

```python
# joining the closing prices of the two datasets
monthly_prices = pd.concat([fb['Close'], sp_500['Close']],
    axis=1)
monthly_prices.columns = ['FB', 'SP500']

# check the head of the dataframe
print(monthly_prices.head())

# calculate monthly returns
monthly_returns = monthly_prices.pct_change(1)
clean_monthly_returns = monthly_returns.dropna(axis=0)

# split dependent and independent variable
X = clean_monthly_returns['SP500']
y = clean_monthly_returns['FB']

# Add a constant to the independent value
X1 = sm.add_constant(X)

# make regression model
model = sm.OLS(y, X1)

# fit model and print results
results = model.fit()
print(results.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                     FB   R-squared:                       0.101
Model:                            OLS   Adj. R-squared:                  0.074
Method:                 Least Squares   F-statistic:                     3.816
Date:                Tue, 19 Feb 2019   Prob (F-statistic):             0.0590
Time:                        10:49:51   Log-Likelihood:                 57.383
No. Observations:                  36   AIC:                            -110.8
Df Residuals:                      34   BIC:                            -107.6
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0203      0.009      2.330      0.026       0.003       0.038
^GSPC          0.5751      0.294      1.953      0.059      -0.023       1.173
==============================================================================
Omnibus:                        0.948   Durbin-Watson:                   2.208
Prob(Omnibus):                  0.623   Jarque-Bera (JB):                0.232
Skew:                          -0.074   Prob(JB):                        0.891
Kurtosis:                       3.364   Cond. No.                         34.9
==============================================================================
```

Figure 1: Summary of the OLS fit: our regression model gives a Beta value of 0.5751 which is very close to the quoted Beta of 0.58