Introduction to Mathematical Finance

Exercise sheet 3

Please hand in your solutions until Friday, 13.03.2020, 13:00 into Bálint Gersey's box next to HG G 53.2.

Exercise 3.1 Let H be a payoff at time T and Ψ a consistent price system.

(a) Show that if H is attainable, then $R^H \in \mathrm{Span}(R^{D^\ell}, 0 \le \ell \le N)$.

(b) Suppose that Q is the EMM associated to Ψ and H is attainable. Also assume that D^0 is a bond with interest rate r. Compute $E_Q[R^H]$.

Exercise 3.2 Let Ψ be a consistent price system and H a payoff at time T. Suppose that the market is arbitrage-free and that H is attainable. Also assume that D^0 is a bond with interest rate r. Show that under any probability measure $Q \sim P$,

$$E_Q[R^H] - r = -\mathrm{Cov}_Q\left(\frac{dP^*}{dQ}, R^H\right),$$

where $P^* \in \mathbb{P}$ is the EMM associated to Ψ and Cov_Q denotes the covariance with respect to Q. *Hint: use the result from Exercise 3.1 (b)*

Note: Proving CAPM-type relation is much simpler if H is attainable.

Exercise 3.3 Recall the setup in Exercise 2.4, where

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix}$$

for some $r > -1$, u, m and d with $u \ge m \ge d$ and $u > r > d$. Denote by $\mathbb{P}_a$ the set of all martingale measures Q which are absolutely continuous with respect to P, i.e., $Q \ll P$.

(a) Show that $\mathbb{P}_a = \overline{\mathbb{P}}$. Here we identity $\mathbb{P}$ with a subset of $\mathbb{R}^K_+ = \mathbb{R}^3_+$ and denote by $\overline{}$ the closure in $\mathbb{R}^K$.

(b) Use (a) to show that for any random variable X,

$$\sup_{Q \in \mathbb{P}} E_Q[X] = \sup_{Q \in \mathbb{P}_a} E_Q[X].$$

(c) Show that for any payoff H, the supremum

$$\sup_{Q \in \mathbb{P}_a} E_Q\left[\frac{H}{D^0}\right]$$

is attained in some $Q \in \mathbb{P}_a$. Does this imply that the market is complete?

Exercise 3.4 Let

$$\pi = \begin{pmatrix} 1 \\ 1000 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1.1 & 1200 \\ 1.1 & 1100 \\ 1.1 & 800 \end{pmatrix}.$$

This is similar to the example with the gold market from the lecture, but now with three possible outcomes. Denote by H the payoff of a put option with strike $K = 900$, i.e.,

$$H = (900 - D^1)^+ = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}.$$

(a) Find

$$\sup_{Q \in \mathbb{P}(D^0)} E_Q\left[\frac{H}{D^0}\right].$$

(b) Find

$$\inf\{\vartheta \cdot \pi : \mathcal{D}\vartheta \geq H\}.$$

(c) Construct a market with $\mathbb{P}_a \neq \overline{\mathbb{P}}$, where we use the notation from Exercise 3.3.

Exercise 3.5 CAPM Analysis: Calculating stock Beta as a Regression. Recall that CAPM is a model that prices securities by examining the relationship between expected returns and risk. More precisely the model states that the return of a risky asset is given by

$$E[R_i] = r + \beta_i(E[R_m] - r) \tag{1}$$

where $E[R_i]$ and $E[R_m]$ are the expected returns of asset i and of the market respectively and r is the risk-free rate of interest. We have seen in the lecture that β_i has a closed from solution given by

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$$

The Beta of an asset is thus a measure of the sensitivity of its returns relative to a market benchmark (usually a market index). We could in principle calculate the Beta of a given asset using the above formula, however in practice it is easier to think of the Beta as the slope of the linear regression (1) and estimate it using Ordinary Least Squares method.

In this exercise we will estimate the Beta of Facebook relative to the S&P500 market index using historical data from 02/11/2014 to 19/11/2017. The data can be downloaded from the course web page ('FB.csv' and 'SP500.csv'). Your task is to complete the below Python code in order to perform an Ordinary Least Squares Regression with Statsmodel. If your code works well, you should get a Beta value close to 0.58 which is the Beta value of Facebook quoted on Yahoo Finance on the 19/11/2017.

```
# import libraries
import pandas as pd
import statsmodels.api as sm

'''
Download monthly prices of Facebook and S&P 500 index from
    2014 to 2017
CSV file downloaded from Yahoo File
start period: 02/11/2014
end period: 30/11/2017
period format: DD/MM/YEAR
'''

# Step 1: Use pandas read_csv method to load the two csv
    files downloaded from the course web page

fb = # TO DO #
sp_500 = # TO DO #

# joining the closing prices of the two datasets
monthly_prices = pd.concat([fb['Close'], sp_500['Close']],
    axis=1)
monthly_prices.columns = ['FB', 'SP500']

# check the head of the dataframe
print(monthly_prices.head())

# calculate monthly returns
monthly_returns = # TO DO #
clean_monthly_returns = monthly_returns.dropna(axis=0)

# split dependent and independent variable
X = # TO DO #
y = # TO DO #

# Add a constant to the independent value
X1 = sm.add_constant(X)
```

```python

# make regression model
model = # TO DO #

# fit model and print results
results = # TO DO #
print(results.summary())
```

Make sure you understand the summary of the linear regression and that in particular you can find the corresponding Beta value.