Introduction to Mathematical Finance

Solution sheet 3

Solution 3.1

(a) Since H is attainable, there exists some $\vartheta \in \mathbb{R}^N$ such that $H = \mathcal{D}\vartheta$. We thus have

$$R^H = \frac{H}{\Psi(0,H)} - 1 = \frac{\mathcal{D}\vartheta - \vartheta \cdot \pi}{\vartheta \cdot \pi} = \sum_{\ell=0}^{N} \frac{\vartheta^\ell (D^\ell - \pi^\ell)}{\pi^\ell} \frac{\pi^\ell}{\vartheta \cdot \pi} = \sum_{\ell=0}^{N} \frac{\vartheta^\ell \pi^\ell}{\vartheta \cdot \pi} R^{D^\ell}.$$

where in the first equality we have used the definition of the return of H.

(b) By Theorem I.5.2, we know that there exists a bijection between set $\mathbb{P}$ of all EMMs Q and the set of all consistent price systems Ψ on $\mathcal{C}$ given by $E_Q[H] = \Psi(0, D^0 H) = (1+r)\Psi(0,H)$ (note that in the last equality we have used the linearity of the price systems Ψ and the fact that D^0 is a bond with interest rate r). Moreover since $R^H = \frac{H - \Psi(0,H)}{\Psi(0,H)}$, we have

$$E_Q[R^H] = \frac{E_Q[H] - \Psi(0,H)}{\Psi(0,H)} = r.$$

Solution 3.2 Set $Z^* := \frac{dP^*}{dQ}$. We compute, using $E_Q[Z^*] = 1$ and $E_{P^*}[R^H] = r$ from Exercise 3.1(b),

$$\begin{aligned}
\mathrm{Cov}_Q(Z^*, R^H) &= E_Q[Z^* R^H] - E_Q[Z^R] E_Q[R^H] \\
&= E_{P^*}[R^H] - E_Q[R^H] \\
&= r - E_Q[R^H] .
\end{aligned}$$

This completes the proof.

Note: Proving CAPM-type relation is much simpler if H is attainable.

Solution 3.3

(a) Recall from the solution of Exercise 2.4 that

$$\mathbb{P}_a = \left\{ \left(\frac{(r-d) - (m-d)\lambda}{u-d}, \lambda, \frac{(u-r) - (u-m)\lambda}{u-d} \right) : \lambda \in \left[0, \min\left\{ \frac{r-d}{m-d}, \frac{u-r}{u-m} \right\}\right] \right\},$$

which is equal to $\overline{\mathbb{P}}$.

Alternative solution to (a). Because $u > r > d$, $\mathbb{P} \neq \varnothing$. Take $P^* \in \mathbb{P}$ and any $Q \in \mathbb{P}_a$. Consider $Q_\varepsilon := \varepsilon P^* + (1-\varepsilon)Q$ and observe that

- Q_ε is a martingale measure since both P^* and Q are martingale measures
- Q_ε and P are equivalent since $P^* \sim P$ and $Q << P$

Taking the limit as $\epsilon \to 0$, we get that $Q \in \overline{\mathbb{P}}$.

(b) Let R be an element in $\mathbb{P}_a$. Let Q be an arbitrary element in $\mathbb{P}$ and construct $Q^\varepsilon = \varepsilon Q + (1-\varepsilon)R$. Then $Q^\varepsilon \in \mathbb{P}$ for all $\varepsilon \in (0,1]$ by construction, and

$$\lim_{\varepsilon \searrow 0} E_{Q^\varepsilon}[X] = \lim_{\varepsilon \searrow 0} \varepsilon E_Q[X] + (1-\varepsilon)E_R[X] = E_R[X],$$

implying that

$$\sup_{Q \in \mathbb{P}} E_Q[X] \geq \sup_{Q \in \mathbb{P}_a} E_Q[X].$$

The converse inequality inequality is trivial.

(c) By (a), $\mathbb{P}_a$ is closed and bounded, so $\mathbb{P}_a$ is compact. This uses again that we identify $\mathbb{P}$ and $\mathbb{P}_a$ with subset of $\mathbb{R}_+^K$. Also the mapping

$$\begin{aligned} \ell : \mathbb{P}_a &\to \mathbb{R}, \\ Q &\mapsto E_Q[X] \end{aligned}$$

is linear and hence continuous because $\mathbb{P}_a$ is finite-dimensional here. Thus the supremum is in fact attained.

Clearly the above result does not depend on the particular values of r, u, m, d. All we need is $u > r > d$ to have $\mathbb{P} \neq \varnothing$. So we can adjust $\mathcal{D}$ to obtain an incomplete market by taking $u > m > d$, while the conclusion of (c) is still true.

Solution 3.4 Note that $\mathcal{D}$ is of the form

$$\mathcal{D} = \begin{pmatrix} \pi^0(1+r) & \pi^1(1+u) \\ \pi^0(1+r) & \pi^1(1+m) \\ \pi^0(1+r) & \pi^1(1+d) \end{pmatrix},$$

with $r = 0.1$, $u = 0.2$, $m = 0.1$ and $d = -0.2$.

(a) From Exercise 2.4 we know that

$$\mathbb{P} = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d} \right) : \lambda \in \left(0, \min\left\{ \frac{r-d}{m-d}, \frac{u-r}{u-m} \right\}\right) \right\}.$$

Set $q_i = Q[\{\omega_i\}]$ for $i \in \{1,2,3\}$. Since $E_Q[H/D^0] = \sum_{i=1}^{3} q_i H(\omega_i)/D^0 = \frac{100 q_3}{1.1}$, which is decreasing in λ, we find the supremum by setting $\lambda = 0$ to obtain

$$\sup_{Q \in \mathbb{P}} E_Q\left[\frac{H}{D^0}\right] = \frac{100}{1.1}\frac{1}{4} = \frac{25}{1.1} = 22.73.$$

(b) Writing down the condition $\mathcal{D}\vartheta \geq H$, we obtain the optimisation problem

$$\begin{aligned} \min \quad & \vartheta^0 + \pi^1\vartheta^1 \\ \text{s.t.} \quad & 11\vartheta^0 + 12\pi^1\vartheta^1 \geq 0, \\ & \vartheta^0 + \pi^1\vartheta^1 \geq 0, \\ & 11\vartheta^0 + 8\pi^1\vartheta^1 \geq 10H_3 = 1000. \end{aligned}$$

Note that the solution is found at an extreme point of the set of feasible solutions, and that the second condition cannot be satisfied with equality without violating the other two. Therefore, solving the outer two inequalities with equality gives

$$\vartheta^s = \begin{pmatrix} \frac{30}{11} \\ -\frac{5}{2\pi^1} \end{pmatrix} H_3$$

as a solution to the optimisation problem, and

$$\pi \cdot \vartheta^s = \frac{25}{1.1}.$$

(c) Extend the market with the asset H at the price $\pi_s(H)$. Denote by $\tilde{\mathbb{P}}$ (resp. $\tilde{\mathbb{P}}_a$) the set of all equivalent (resp. all absolutely continuous) martingale measures with the numéraire D^0 in the extended market. Then, from the characterisation of martingale measures to the original market, we conclude that

$$\tilde{\mathbb{P}}_a = \left\{\left(\frac{3}{4}, 0, \frac{1}{4}\right)\right\},$$

(this is the measure corresponding to $\lambda = 0$) which is of course not an *equivalent* martingale measure. Hence, $\tilde{\mathbb{P}} = \varnothing$ and $\tilde{\mathbb{P}}_a \neq \overline{\tilde{\mathbb{P}}}$.

A simpler example is given by the market

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

where the only measure satisfying the martingale property is identified by $(0,1)$, which is not equivalent to P.

Solution 3.5

```python
# import libraries
import pandas as pd
import statsmodels.api as sm

'''
Download monthly prices of Facebook and S&P 500 index from 2014 to 2017
CSV file downloaded from Yahoo File
start period: 02/11/2014
end period: 30/11/2017
period format: DD/MM/YEAR
'''
fb = pd.read_csv('FB.csv', parse_dates=True, index_col='Date',)
sp_500 = pd.read_csv('SP500.csv', parse_dates=True, index_col='Date')

# joining the closing prices of the two datasets
monthly_prices = pd.concat([fb['Close'], sp_500['Close']], axis=1)
monthly_prices.columns = ['FB', 'SP500']

# check the head of the dataframe
print(monthly_prices.head())

# calculate monthly returns
monthly_returns = monthly_prices.pct_change(1)
clean_monthly_returns = monthly_returns.dropna(axis=0)

# split dependent and independent variable
X = clean_monthly_returns['SP500']
y = clean_monthly_returns['FB']

# Add a constant to the independent value
X1 = sm.add_constant(X)

# make regression model
model = sm.OLS(y, X1)

# fit model and print results
results = model.fit()
print(results.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                     FB   R-squared:                       0.101
Model:                            OLS   Adj. R-squared:                  0.074
Method:                 Least Squares   F-statistic:                     3.816
Date:                Tue, 19 Feb 2019   Prob (F-statistic):             0.0590
Time:                        10:49:51   Log-Likelihood:                 57.383
No. Observations:                  36   AIC:                            -110.8
Df Residuals:                      34   BIC:                            -107.6
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0203      0.009      2.330      0.026       0.003       0.038
^GSPC          0.5751      0.294      1.953      0.059      -0.023       1.173
==============================================================================
Omnibus:                        0.948   Durbin-Watson:                   2.208
Prob(Omnibus):                  0.623   Jarque-Bera (JB):                0.232
Skew:                          -0.074   Prob(JB):                        0.891
Kurtosis:                       3.364   Cond. No.                         34.9
==============================================================================
```

Figure 1: Summary of the OLS fit: our regression model gives a Beta value of 0.5751 which is very close to the quoted Beta of 0.58