Introduction to Mathematical Finance

Exercise sheet 4

Please hand in your solutions until Friday, 20/03/2020, 13:00 into Bálint Gersey's box next to HG G 53.2.

Exercise 4.1 Let $(\mathcal{D}, \pi)$ be an arbitrage-free market with numéraire. You can assume that in such a market, for any payoff H, there exists a strategy ϑ^s which attains the infimum in the definition of $\pi_s(H)$.

Consider a payoff H which is not attainable in $\mathcal{D}$ and $\pi_s(H)$ the seller's price for H, i.e.,

$$\pi_s(H) = \inf\{\vartheta \cdot \pi : \vartheta \in \mathbb{R}^N \text{ with } \mathcal{D}\vartheta \geq H\}.$$

Denote by $(\mathcal{D}^e, \pi^e)$ the extended market $(\mathcal{D}, H, \pi, \pi_s(H))$.

(a) Show that $(\mathcal{D}^e, \pi^e)$ always admits an arbitrage opportunity of the first kind.

(b) Show that $(\mathcal{D}^e, \pi^e)$ does not admit an arbitrage opportunity of the second kind.

(c) *(Difficult bonus question)* Let $(\mathcal{D}, \pi)$ be an arbitrage free market having a numéraire and let H be any payoff. Show that we can always find a strategy ϑ^s which attains the infimum in the definition of $\pi_s(H)$.

Solution 4.1

(a) Denote by ϑ^s the strategy in $(\mathcal{D}, \pi)$ which attains the infimum in the definition of $\pi_s(H)$. Consider the strategy in $(\mathcal{D}^e, \pi^e)$ given by

$$\vartheta^a = \begin{pmatrix} \vartheta^s \\ -1 \end{pmatrix}.$$

Then, by construction of ϑ^s,

$$\vartheta^a \cdot \pi^e = \vartheta^s \cdot \pi - \pi_s(H) = 0$$

and

$$\mathcal{D}^e \vartheta^a = \mathcal{D}\vartheta^s - H \geq 0,$$

with strict inequality in some outcome (or else H would be attainable), showing that ϑ^a is an arbitrage opportunity of the first kind in (D^e, π^e).

(b) Let $N+1$ be the index of the asset with payoff H in the extended market and denote again by ϑ^s the strategy in $(\mathcal{D}, \pi)$ which attains the infimum in the definition of $\pi_s(H)$. Suppose there exists an arbitrage opportunity ϑ^a of the second kind in $(\mathcal{D}^e, \pi^e)$. Denote by ϑ^{a-} the vector

$$\begin{pmatrix} \vartheta_1^a \\ \vdots \\ \vartheta_N^a \end{pmatrix}.$$

Define ϑ according to

$$\vartheta = \vartheta^{a-} + \vartheta^a_{N+1}\vartheta^s.$$

Then

$$\begin{aligned} \vartheta^a \cdot \pi^e &= \vartheta^{a-} \cdot \pi + \vartheta^a_{N+1}\pi_s(H) \\ &= \vartheta^{a-} \cdot \pi + \vartheta^a_{N+1}\vartheta^s \cdot \pi \\ &= \vartheta \cdot \pi \end{aligned}$$

In the second equality, we used that ϑ^s the strategy in $(\mathcal{D}, \pi)$ from which attains the infimum in the definition of $\pi_s(H)$ and the last equality comes from the definition of ϑ. Since by assumption ϑ^a is an arbitrage of the second kind in (D^e, π^e), we have $\vartheta \cdot \pi < 0$.

If $\vartheta^a_{N+1} \geq 0$, then

$$\begin{aligned} \mathcal{D}\vartheta &= \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}\mathcal{D}\vartheta^s \\ &\geq \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}H \\ &= \mathcal{D}^e\vartheta^a \geq 0 \end{aligned}$$

The first equality comes from the definition of ϑ; in the second line we used that ϑ^s the strategy in $(\mathcal{D}, \pi)$ from which attains the infimum in the definition of $\pi_s(H)$ and so in particular $\mathcal{D}\vartheta^s \geq H$; the last equality is a consequence of the definition of ϑ^{a-}. This implies that $(\mathcal{D}, \pi)$ has an arbitrage of the second kind. However, since that market is arbitrage-free, we reach a contradiction.

If, on the other hand, $\vartheta^a_{N+1} < 0$, then

$$\begin{aligned} 0 \leq \mathcal{D}^e\vartheta^a &= \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}H \\ &= \mathcal{D}\vartheta^{a-} - |\vartheta^a_{N+1}|H \end{aligned}$$

The first inequality holds because ϑ^a is an arbitrage of the second kind in (D^e, π^e); the equality on the first line follows from the definition of $\mathcal{D}^e$ and ϑ^a; and the equality of the second line uses the assumption $\vartheta^a_{N+1} < 0$. This implies that

$$|\vartheta^a_{N+1}|H \leq \mathcal{D}\vartheta^{a-}, \text{ hence } H \leq \mathcal{D}\frac{\vartheta^{a-}}{|\vartheta^a_{N+1}|}.$$

Thus,

$$\vartheta^a \cdot \pi^e = |\vartheta^a_{N+1}| \left(\frac{\vartheta^{a-}}{|\vartheta^a_{N+1}|} \cdot \pi - \pi_s(H) \right) \geq 0$$

by the definition of $\pi_s(H)$. This contradicts the fact that ϑ^a is an arbitrage of the second kind in $(\mathcal{D}^e, \pi^e)$.

Conclusion: We must have $\vartheta^a_{N+1} \geq 0$ but in that case we have find a contradiction so there cannot exist an arbitrage of the second kind in (D^e, π^e).

Exercise 4.2 Let H be a payoff at time T.

(a) Assume the binomial model (Exercise 2.3) with $d < r < u$. Suppose that $H = f(D^1)$ for some convex function $f \geq 0$. Compute $\pi_s(H)$.

(b) Now assume only that the market is arbitrage-free. Let

$$\pi = \begin{pmatrix} \pi^0 \\ \pi^1 \end{pmatrix}.$$

Suppose that $H = f(D^1)$ for some convex function $f \geq 0$. Show the inequalities:

$$\pi_b(H) \geq \frac{f(\pi^1(1+r))}{1+r} \text{ and } \pi_s(H) \leq \frac{f(0)}{1+r} + \lim_{x \uparrow \infty} \frac{f(x)}{x} \pi^1.$$

Hint: First prove the second inequality with lim sup. Then show the limit exists.

Solution 4.2

(a) By Exercise 2.3, a binomial market with $d < r < u$ is arbitrage free and complete. Indeed in Exercise 2.3 we found a unique EMM Q (under the assumption $d < r < u$) given by

$$q_u = \frac{r-d}{u-d}, \quad q_d = \frac{u-r}{u-d}$$

and so

- by the First Fundamental Theorem of Asset Pricing (Theorem I.4.3), the market is arbitrage-free (since $\mathbb{P} \neq \emptyset$)
- by the Second Fundamental Theorem of Asset Pricing (Theorem I.4.5), the market is complete (since $|\mathbb{P}| = 1$)

We use Theorem I.7.2 to compute $\pi_s(H)$:

$$\pi_s(H) = \sup_{Q \in \mathbb{P}} E_Q \left[\frac{H}{D^0} \right] = E_Q \left[\frac{H}{1+r} \right] = \frac{f(1+u)}{1+r} \frac{r-d}{u-d} + \frac{f(1+d)}{1+r} \frac{u-r}{u-d}.$$

(b) Again since the market is arbitrage-free, we can find $Q \in \mathbb{P}$. Then we first estimate, using Jensen's inequality, that

$$\begin{aligned} E_Q \left[\frac{H}{D^0} \right] &= \frac{E_Q[f(D^1)]}{1+r} \\ &\geq \frac{f(E_Q[D^1])}{1+r} \\ &= \frac{f(\pi^1(1+r))}{1+r}. \end{aligned}$$

Taking the infimum over all $Q \in \mathbb{P}$ and using Theorem I.7.2 yields the first inequality.

Because f is nonnegative, $L := \limsup_{x\to\infty} \frac{f(x)}{x} \in [0, +\infty]$. For $L = +\infty$, the second inequality is trivial. So assume $L < \infty$. Using convexity of f to write

$$f\left(\frac{x}{y}y + (1-\frac{x}{y})0\right) \leq \frac{x}{y}f(y) + (1-\frac{x}{y})f(0)$$

for $0 \leq x \leq y$, we obtain

$$\frac{f(x)-f(0)}{x} \leq \frac{f(y)-f(0)}{y}, \forall 0 \leq x \leq y.$$

Since Ω is finite, we have $y \geq D^1$ P-a.s. for sufficiently large y, and therefore

$$\begin{aligned} E_Q[f(D^1)] &= f(0) + E_Q[f(D^1) - f(0)] \\ &\leq f(0) + E_Q[\frac{f(y)-f(0)}{y} D^1] \\ &\leq f(0) + E_Q[D^1] \limsup_{y\uparrow\infty} \frac{f(y)-f(0)}{y} \\ &= f(0) + \pi^1(1+r)L. \end{aligned}$$

Divinding on both sides by $(1+r)$ yields

$$E_Q\left[\frac{f(D^1)}{D^0}\right] \leq \frac{f(0)}{1+r} + \limsup_{x\to\infty} \frac{f(x)}{x}\pi^1.$$

Taking the supremum over $Q \in \mathbb{P}$ yields the second inequality.

To see that $\lim_{x\to\infty} f(x)/x$ exists, observe that

$$\frac{f(x)}{x} = \frac{f(x)-f(0)}{x} + \frac{f(0)}{x} =: I_1(x) + I_2(x).$$

We saw that $I_1(x)$ is increasing, so $\lim_{x\to\infty} I_1(x)$ exists in $[0, +\infty]$. Also, $\lim_{x\to\infty} I_2(x) = 0$, we obtain that $\lim_{x\to\infty} f(x)/x$ exists.

Exercise 4.3 Consider an arbitrage-free market with a single risky asset D^1. Assume D^0 is a bond with interest rate $r > -1$. Set

$$\pi = \begin{pmatrix} 1 \\ \pi^1 \end{pmatrix}.$$

Recall that a *call option* on D^1 with strike K is defined by $H^c := (D^1 - K)^+$ and a put option with strike K is defined by $H^p := (K - D^1)^+$.

(a) Suppose that the market is complete. Show that the arbitrage-free prices $\pi(H^c)$ and $\pi(H^p)$ of H^c and H^p, respectively, are related by

$$\pi(H^c) - \pi(H^p) = \pi^1 - \frac{K}{1+r}.$$

This relation is known as the *put-call parity.*

(b) Show that

$$\left(\pi^1 - \frac{K}{1+r}\right)^+ \leq \pi_b(H^c) \leq \pi_s(H^c) \leq \pi^1. \qquad (*)$$

Derive the analogous bounds for H^p.

(c) Assume $r \geq 0$. Compare $(\pi^1 - K)^+$ and $\pi_b(H^c)$. The first quantity is also known as the *intrinsic value* of the call option. Can you give a financial interpretation for the result of this comparison? Do we have a similar situation for the put option?

(d) *(Bonus)* Writing P_0, C_0, S_0 and B_0 for the initial price of the put, call, stock and bond respectively, the put-call parity formula can be rewritten as

$$P_0 - C_0 = B_0 K - S_0$$

This is the equation of a line. Using the programming language of your choice, verify the put-call parity formula on historical prices. To do this, you are asked to

- plot $P_0 - C_0$ versus K, where $t = 0$ corresponds to 23 October 2017 and $t = T$ is 17 November 2017, and the underlying asset is the $S\&P500$ index. You can take the price of the calls and puts to be the last traded price on the day (as opposed to bid or ask price). You can find all data needed on yahoo finance.
- perform a linear regression of the response variable $P_0 - C_0$ against the predictor K. What are the obtained coefficients of the regression? Perform a goodness of fit analysis to judge the quality of your fitted model.

Solution 4.3

(a) **1st solution: using EMM**

Since the market is complete, there is a unique EMM Q. We observe $H^c - H^p = D^1 - K$. Then we discount by D^0 and take expectation under Q to obtain

$$\pi(H^c) - \pi(H^p) = E_Q\left[\frac{D^1}{D^0}\right] - E_Q\left[\frac{K}{D^0}\right] = \pi^1 - \frac{K}{1+r}$$

2nd solution: using a replication argument

The idea is to find the initial price of the put option by finding an investment strategy consisting of investments in the bond D^0, the risky asset D^1 and the call option C, that replicates the payoff of the put option. In the same way as in the first solution, we have $H^p = K - D^1 + H^c$ and thus one can easily see that the strategy consisting of

- being long K units of bond
- being short one unit of D^1
- being long one unit of C

replicates the terminal payoff of the put option. By no arbitrage, the initial value of the put option must coincide with the initial value of our replicating portfolio and hence

$$\pi(H^p) = \frac{K}{1+r} - \pi^1 + \pi(H^c)$$

(b) The second inequality is proved in Lemma I.7.1. The third inequality is immediate by $H^c \leq D^1$ and the no arbitrage assumption. Let $Q \in \mathbb{P}$. Observe that $x \mapsto (x-K)^+$ is convex. We use Jensen's inequality to get the first inequality via

$$\begin{aligned} E_Q\left[\frac{H^c}{D^0}\right] &= E_Q\left[\frac{(D^1-K)^+}{D^0}\right] = E_Q\left[\left(\frac{D^1-K}{D^0}\right)^+\right] \\ &\geq \left(E_Q\left[\frac{D^1-K}{D^0}\right]\right)^+ = \left(\pi^1 - \frac{K}{1+r}\right)^+ \end{aligned}$$

Moreover by Theorem I.7.2, we have $\pi_b(H^c) = \inf_{Q\in\mathbb{P}} E_Q[\frac{H^c}{D^0}]$ and so taking the infimum in the above inequality gives

$$\pi_b(H^c) \geq \left(\pi^1 - \frac{K}{1+r}\right)^+$$

For the put option H^p, note that $H^p \leq K$, and this gives

$$\pi_s(H^p) \leq E_Q\left[\frac{K}{D^0}\right] = \frac{K}{1+r}.$$

We use again Jensen's inequality to bound

$$\pi_b(H^p) \geq \inf_{Q\in\mathbb{P}} \left(E_Q\left[\frac{K - D^1}{D^0}\right]\right)^+ = \left(\frac{K}{1+r} - \pi^1\right)^+.$$

(c) Since $r \geq 0$, we obtain by the previous question

$$\pi_s(H^c) \geq (\pi^1 - K)^+.$$

This inequality says that the value of the right to buy D^1 at time 0 is less than the arbitrage-free price of H^c. In other words, the *time value* of a call option is positive.

However, for the put option, we do not have such a bound unless $r \leq 0$.

(d) Your plot should look like

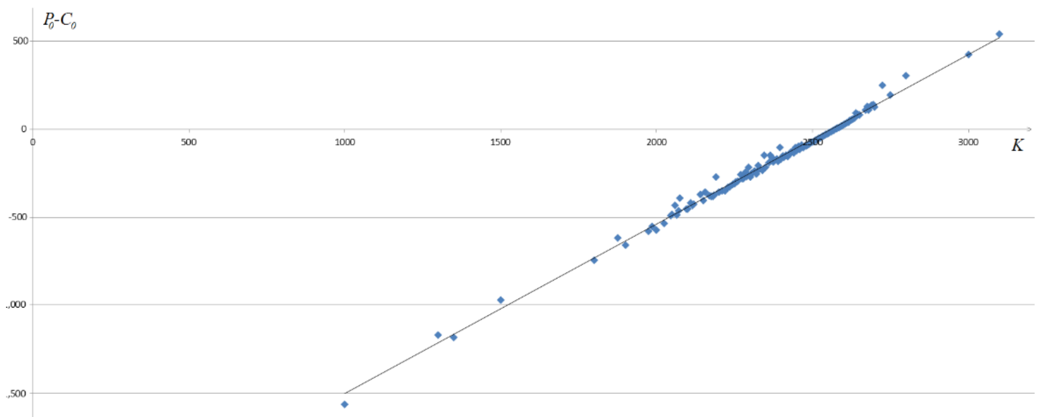
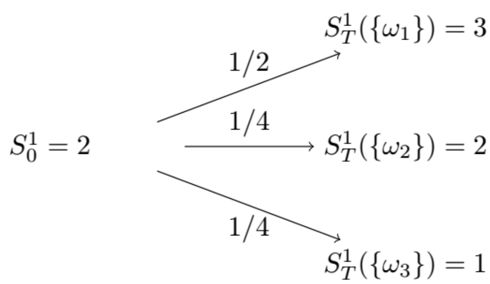


FIGURE 1. A plot of $P_0 - C_0$ versus K, where $t = 0$ corresponds to 23 October 2017 and and $t = T$ is 17 November 2017, and the underlying asset is the S&P 500 index with $S_0 = 2,573.82$. The price of the calls and puts is taken to be the last traded price on the day (as opposed to the bid or ask price). All data is taken from `https://uk.finance.yahoo.com`.

From $P_0 - C_0 = B_0K - S_0$, one can directly see that the intercept coefficient should be close to $-S_0$ and the coefficient for K should be close to B_0. For the statistical interpretation of what "close" means and how to analyze the output of a linear regression, as well as how to decide of the quality of the fitted model, you are encouraged to read Chapter 3 of "An Introduction to Statistical Learning" by R. Tibshirani et al.

The above plot was taken from lecture notes of the Advanced Financial Models taught at the University of Cambridge by Mike Tehranchi in 2017.

Exercise 4.4 Consider a trinomial two-asset model. The first asset is a risk-free bond with initial value $S_0^0 = 1$ and the second asset is a risky stock with initial value $S_0^1 = 2$ and whose evolution under the real world measure P is given by the following tree:



We also suppose that the spot interest rate is $r = 0$.

(a) Find all risk-neutral measures for this model.

Now introduce a call option on the risky asset with strike $K = 2$ and maturity T.

(b) What is the terminal payoff H of this contingent claim?

(c) Find the least expensive super replicating portfolio, i.e. the portfolio that attains the infimum in the definition of $\pi_s(H)$.

(d) Find the most expensive sub-replicating portfolio.

Solution 4.4

(a) An equivalent martingale measure solves $E_Q[S_1^1] = S_0^1$ (no discounting is needed since $r = 0$). By Lemma I.4.1, we can identify the measure Q with a vector $q = (q_1, q_2, q_3)^\top \in \mathbb{R}^3_{++}$ where $q_1 = Q(S_1^1 = 3)$, $q_1 = Q(S_1^1 = 2)$ and $q_1 = Q(S_1^1 = 1)$. Q being an EMM, give the following equations:

$$\begin{cases} 3q_1 + 2q_2 + q_3 = 2 \\ q_1 + q_2 + q_3 = 1 \\ q_1, q_2, q_2 > 0 \end{cases}$$

This gives that the set of all EMMs is given by

$$\mathbb{P} = \{(q_1, 1 - 2q_1, q_1) \quad | \quad 0 < q_1 < 1/2\}$$

(b) The terminal payoff of the call option is $(S_T - K)^+$.

(c) Denoting θ^0 and θ^1 the holding in the bond and the stock, to super-replicate the payout H, we are asked to minimize $\theta^0 + 2\theta^1$ subject to the constraints

$$\begin{cases} \theta^0 + 3\theta^1 \geq 1 \\ \theta^0 + 2\theta^1 \geq 0 \\ \theta^0 + \theta^1 \geq 0 \end{cases}$$

By Theorem I.7.2 and the bonus question of Problem 1, we now that there exist a vector $\theta^* = (\theta^{0*}, \theta^{1*})^\top$ such that

$$\theta^{0*} S_0^0 + \theta^{1*} S_0^1 = \sup_{Q \in \mathbb{P}} E_Q[H] = \sup_{0<q_1<1/2} [q_1 + (1 - 2q_1) \cdot 0 + q_1 \cdot 0] = 1/2$$

Unfortunately, the Linear Programming duality principle only tells us how to calculate the seller's price but tells nothing about the strategy that would give that price. However, for this problem, we can easily see that $\theta^* = (-1/2, 1/2)^\top$.

(d) Similarly, the most expensive sub-replication cost is 0 and is attained for $\theta^* = (0, 0)^\top$.