# Introduction to Mathematical Finance

## Exercise sheet 4

*Please hand in your solutions until Friday, 20/03/2020, 13:00 into Bálint Gersey's box next to HG G 53.2.*

**Exercise 4.1** Let $(\mathcal{D}, \pi)$ be an arbitrage-free market with numéraire. You can assume that in such a market, for any payoff $H$, there exists a strategy $\vartheta^s$ which attains the infimum in the definition of $\pi_s(H)$.

Consider a payoff $H$ which is not attainable in $\mathcal{D}$ and $\pi_s(H)$ the seller's price for $H$, i.e.,

$$\pi_s(H) = \inf\{\vartheta \cdot \pi : \vartheta \in \mathbb{R}^N \text{ with } \mathcal{D}\vartheta \geq H\}.$$

Denote by $(\mathcal{D}^e, \pi^e)$ the extended market $(\mathcal{D}, H, \pi, \pi_s(H))$.

(a) Show that $(\mathcal{D}^e, \pi^e)$ always admits an arbitrage opportunity of the first kind.

(b) Show that $(\mathcal{D}^e, \pi^e)$ does not admit an arbitrage opportunity of the second kind.

(c) *(Difficult bonus question)* Let $(\mathcal{D}, \pi)$ be an arbitrage free market having a numéraire and let $H$ be any payoff. Show that we can always find a strategy $\vartheta^s$ which attains the infimum in the definition of $\pi_s(H)$.

**Exercise 4.2** Let $H$ be a payoff at time $T$.

(a) Assume the binomial model (Exercise 2.3) with $d < r < u$. Suppose that $H = f(D^1)$ for some convex function $f \geq 0$. Compute $\pi_s(H)$.

(b) Now assume only that the market is arbitrage-free. Let

$$\pi = \begin{pmatrix} \pi^0 \\ \pi^1 \end{pmatrix}.$$

Suppose that $H = f(D^1)$ for some convex function $f \geq 0$. Show the inequalities:

$$\pi_b(H) \geq \frac{f(\pi^1(1+r))}{1+r} \text{ and } \pi_s(H) \leq \frac{f(0)}{1+r} + \lim_{x\uparrow\infty} \frac{f(x)}{x}\pi^1.$$

*Hint:* First prove the second inequality with $\limsup$. Then show the limit exists.

**Exercise 4.3** Consider an arbitrage-free market with a single risky asset $D^1$. Assume $D^0$ is a bond with interest rate $r > -1$. Set

$$\pi = \begin{pmatrix} 1 \\ \pi^1 \end{pmatrix}.$$

Recall that a *call option* on $D^1$ with strike $K$ is defined by $H^c := (D^1 - K)^+$ and a put option with strike $K$ is defined by $H^p := (K - D^1)^+$.

(a) Suppose that the market is complete. Show that the arbitrage-free prices $\pi(H^c)$ and $\pi(H^p)$ of $H^c$ and $H^p$, respectively, are related by

$$\pi(H^c) - \pi(H^p) = \pi^1 - \frac{K}{1+r}.$$

This relation is known as the *put-call parity*.

(b) Show that

$$\left(\pi^1 - \frac{K}{1+r}\right)^+ \leq \pi_b(H^c) \leq \pi_s(H^c) \leq \pi^1. \tag{$*$}$$

Derive the analogous bounds for $H^p$.

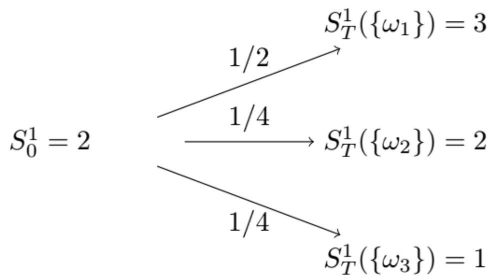
(c) Assume $r \geq 0$. Compare $(\pi^1 - K)^+$ and $\pi_b(H^c)$. The first quantity is also known as the *intrinsic value* of the call option. Can you give a financial interpretation for the result of this comparison? Do we have a similar situation for the put option?

(d) *(Bonus)* Writing $P_0$, $C_0$, $S_0$ and $B_0$ for the initial price of the put, call, stock and bond respectively, the put-call parity formula can be rewritten as

$$P_0 - C_0 = B_0 K - S_0$$

This is the equation of a line. Using the programming language of your choice, verify the put-call parity formula on historical prices. To do this, you are asked to

- plot $P_0 - C_0$ versus $K$, where $t = 0$ corresponds to 23 October 2017 and $t = T$ is 17 November 2017, and the underlying asset is the $S\&P500$ index. You can take the price of the calls and puts to be the last traded price on the day (as opposed to bid or ask price). You can find all data needed on yahoo finance.
- perform a linear regression of the response variable $P_0 - C_0$ against the predictor $K$. What are the obtained coefficients of the regression? Perform a goodness of fit analysis to judge the quality of your fitted model.




**Exercise 4.4** Consider a trinomial two-asset model. The first asset is a risk-free bond with initial value $S_0^0 = 1$ and the second asset is a risky stock with initial value $S_0^1 = 2$ and whose evolution under the real world measure $P$ is given by the following tree:

We also suppose that the spot interest rate is $r = 0$.

(a) Find all risk-neutral measures for this model.

Now introduce a call option on the risky asset with strike $K = 2$ and maturity $T$.

(b) What is the terminal payoff $H$ of this contingent claim?

(c) Find the least expensive super replicating portfolio, i.e. the portfolio that attains the infimum in the definition of $\pi_s(H)$.

(d) Find the most expensive sub-replicating portfolio.