Coordinator
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Introduction to Mathematical Finance

Solution sheet 4

Solution 4.1

(a) Denote by ϑ^s the strategy in $(\mathcal{D}, \pi)$ which attains the infimum in the definition of $\pi_s(H)$. Consider the strategy in $(\mathcal{D}^e, \pi^e)$ given by

$$\vartheta^a = \begin{pmatrix} \vartheta^s \\ -1 \end{pmatrix}.$$

Then, by construction of ϑ^s,

$$\vartheta^a \cdot \pi^e = \vartheta^s \cdot \pi - \pi_s(H) = 0$$

and

$$\mathcal{D}^e \vartheta^a = \mathcal{D}\vartheta^s - H \geq 0,$$

with strict inequality in some outcome (or else H would be attainable), showing that ϑ^a is an arbitrage opportunity of the first kind in (D^e, π^e).

(b) Let $N+1$ be the index of the asset with payoff H in the extended market and denote again by ϑ^s the strategy in $(\mathcal{D}, \pi)$ which attains the infimum in the definition of $\pi_s(H)$. Suppose there exists an arbitrage opportunity ϑ^a of the second kind in $(\mathcal{D}^e, \pi^e)$. Denote by ϑ^{a-} the vector

$$\begin{pmatrix} \vartheta^a_1 \\ \vdots \\ \vartheta^a_N \end{pmatrix}.$$

Define ϑ according to

$$\vartheta = \vartheta^{a-} + \vartheta^a_{N+1}\vartheta^s.$$

Then

$$\begin{aligned} \vartheta^a \cdot \pi^e &= \vartheta^{a-} \cdot \pi + \vartheta^a_{N+1}\pi_s(H) \\ &= \vartheta^{a-} \cdot \pi + \vartheta^a_{N+1}\vartheta^s \cdot \pi \\ &= \vartheta \cdot \pi \end{aligned}$$

In the second equality, we used that ϑ^s the strategy in $(\mathcal{D}, \pi)$ from which attains the infimum in the definition of $\pi_s(H)$ and the last equality comes from the definition of ϑ. Since by assumption ϑ^a is an arbitrage of the second kind in (D^e, π^e), we have $\vartheta \cdot \pi < 0$.

If $\vartheta^a_{N+1} \geq 0$, then

$$\begin{aligned}\mathcal{D}\vartheta &= \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}\mathcal{D}\vartheta^s \\ &\geq \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}H \\ &= \mathcal{D}^e\vartheta^a \geq 0\end{aligned}$$

The first equality comes from the definition of ϑ; in the second line we used that ϑ^s the strategy in $(\mathcal{D}, \pi)$ from which attains the infimum in the definition of $\pi_s(H)$ and so in particular $\mathcal{D}\vartheta^s \geq H$; the last equality is a consequence of the definition of ϑ^{a-}. This implies that $(\mathcal{D}, \pi)$ has an arbitrage of the second kind. However, since that market is arbitrage-free, we reach a contradiction.

If, on the other hand, $\vartheta^a_{N+1} < 0$, then

$$\begin{aligned}0 \leq \mathcal{D}^e\vartheta^a &= \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}H \\ &= \mathcal{D}\vartheta^{a-} - |\vartheta^a_{N+1}|H\end{aligned}$$

The first inequality holds because ϑ^a is an arbitrage of the second kind in (D^e, π^e); the equality on the first line follows from the definition of $\mathcal{D}^e$ and ϑ^a; and the equality of the second line uses the assumption $\vartheta^a_{N+1} < 0$. This implies that

$$|\vartheta^a_{N+1}|H \leq \mathcal{D}\vartheta^{a-}, \text{ hence } H \leq \mathcal{D}\frac{\vartheta^{a-}}{|\vartheta^a_{N+1}|}.$$

Thus,

$$\vartheta^a \cdot \pi^e = |\vartheta^a_{N+1}| \left(\frac{\vartheta^{a-}}{|\vartheta^a_{N+1}|} \cdot \pi - \pi_s(H) \right) \geq 0$$

by the definition of $\pi_s(H)$. This contradicts the fact that ϑ^a is an arbitrage of the second kind in $(\mathcal{D}^e, \pi^e)$.

Conclusion: We must have $\vartheta^a_{N+1} \geq 0$ but in that case we have find a contradiction so there cannot exist an arbitrage of the second kind in (D^e, π^e).

Solution 4.2

(a) By Exercise 2.3, a binomial market with $d < r < u$ is arbitrage free and complete. Indeed in Exercise 2.3 we found a unique EMM Q (under the assumption $d < r < u$) given by

$$q_u = \frac{r-d}{u-d}, \quad q_d = \frac{u-r}{u-d}$$

and so

- by the First Fundamental Theorem of Asset Pricing (Theorem I.4.3), the market is arbitrage-free (since $\mathbb{P} \neq \emptyset$)

- by the Second Fundamental Theorem of Asset Pricing (Theorem I.4.5), the market is complete (since $|\mathbb{P}| = 1$)

We use Theorem I.7.2 to compute $\pi_s(H)$:

$$\pi_s(H) = \sup_{Q\in\mathbb{P}} E_Q\left[\frac{H}{D^0}\right] = E_Q\left[\frac{H}{1+r}\right] = \frac{f(1+u)}{1+r}\frac{r-d}{u-d} + \frac{f(1+d)}{1+r}\frac{u-r}{u-d}.$$

(b) Again since the market is arbitrage-free, we can find $Q \in \mathbb{P}$. Then we first estimate, using Jensen's inequality, that

$$\begin{aligned} E_Q\left[\frac{H}{D^0}\right] &= \frac{E_Q[f(D^1)]}{1+r} \\ &\geq \frac{f(E_Q[D^1])}{1+r} \\ &= \frac{f(\pi^1(1+r))}{1+r}. \end{aligned}$$

Taking the infimum over all $Q \in \mathbb{P}$ and using Theorem I.7.2 yields the first inequality.

Because f is nonnegative, $L := \limsup_{x\to\infty} \frac{f(x)}{x} \in [0, +\infty]$. For $L = +\infty$, the second inequality is trivial. So assume $L < \infty$. Using convexity of f to write

$$f\left(\frac{x}{y}y + (1-\frac{x}{y})0\right) \leq \frac{x}{y}f(y) + (1-\frac{x}{y})f(0)$$

for $0 \leq x \leq y$, we obtain

$$\frac{f(x)-f(0)}{x} \leq \frac{f(y)-f(0)}{y}, \forall 0 \leq x \leq y.$$

Since Ω is finite, we have $y \geq D^1$ P-a.s. for sufficiently large y, and therefore

$$\begin{aligned} E_Q[f(D^1)] &= f(0) + E_Q[f(D^1) - f(0)] \\ &\leq f(0) + E_Q[\frac{f(y)-f(0)}{y}D^1] \\ &\leq f(0) + E_Q[D^1]\limsup_{y\uparrow\infty}\frac{f(y)-f(0)}{y} \\ &= f(0) + \pi^1(1+r)L. \end{aligned}$$

Divinding on both sides by $(1+r)$ yields

$$E_Q\left[\frac{f(D^1)}{D^0}\right] \leq \frac{f(0)}{1+r} + \limsup_{x\to\infty}\frac{f(x)}{x}\pi^1.$$

Taking the supremum over $Q \in \mathbb{P}$ yields the second inequality.

To see that $\lim_{x\to\infty} f(x)/x$ exists, observe that

$$\frac{f(x)}{x} = \frac{f(x)-f(0)}{x} + \frac{f(0)}{x} =: I_1(x) + I_2(x).$$

We saw that $I_1(x)$ is increasing, so $\lim_{x\to\infty} I_1(x)$ exists in $[0,+\infty]$. Also, $\lim_{x\to\infty} I_2(x) = 0$, we obtain that $\lim_{x\to\infty} f(x)/x$ exists.

Solution 4.3

(a) **1st solution: using EMM**

Since the market is complete, there is a unique EMM Q. We observe $H^c - H^p = D^1 - K$. Then we discount by D^0 and take expectation under Q to obtain

$$\pi(H^c) - \pi(H^p) = E_Q\left[\frac{D^1}{D^0}\right] - E_Q\left[\frac{K}{D^0}\right] = \pi^1 - \frac{K}{1+r}$$

2nd solution: using a replication argument

The idea is to find the initial price of the put option by finding an investment strategy consisting of investments in the bond D^0, the risky asset D^1 and the call option C, that replicates the payoff of the put option. In the same way as in the first solution, we have $H^p = K - D^1 + H^c$ and thus one can easily see that the strategy consisting of

- being long K units of bond
- being short one unit of D^1
- being long one unit of C

replicates the terminal payoff of the put option. By no arbitrage, the initial value of the put option must coincide with the initial value of our replicating portfolio and hence

$$\pi(H^p) = \frac{K}{1+r} - \pi^1 + \pi(H^c)$$

(b) The second inequality is proved in Lemma I.7.1. The third inequality is immediate by $H^c \le D^1$ and the no arbitrage assumption. Let $Q \in \mathbb{P}$. Observe that $x \mapsto (x-K)^+$ is convex. We use Jensen's inequality to get the first inequality via

$$\begin{aligned} E_Q\left[\frac{H^c}{D^0}\right] &= E_Q\left[\frac{(D^1-K)^+}{D^0}\right] = E_Q\left[\left(\frac{D^1-K}{D^0}\right)^+\right] \\ &\ge \left(E_Q\left[\frac{D^1-K}{D^0}\right]\right)^+ = \left(\pi^1 - \frac{K}{1+r}\right)^+ \end{aligned}$$

Moreover by Theorem I.7.2, we have $\pi_b(H^c) = \inf_{Q\in\mathbb{P}} E_Q[\frac{H^c}{D^0}]$ and so taking the infimum in the above inequality gives

$$\pi_b(H^c) \geq \left(\pi^1 - \frac{K}{1+r}\right)^+$$

For the put option H^p, note that $H^p \leq K$, and this gives

$$\pi_s(H^p) \leq E_Q\left[\frac{K}{D^0}\right] = \frac{K}{1+r}.$$

We use again Jensen's inequality to bound

$$\pi_b(H^p) \geq \inf_{Q\in\mathbb{P}} \left(E_Q\left[\frac{K-D^1}{D^0}\right]\right)^+ = \left(\frac{K}{1+r} - \pi^1\right)^+.$$

(c) Since $r \geq 0$, we obtain by the previous question

$$\pi_s(H^c) \geq (\pi^1 - K)^+.$$

This inequality says that the value of the right to buy D^1 at time 0 is less than the arbitrage-free price of H^c. In other words, the *time value* of a call option is positive.

However, for the put option, we do not have such a bound unless $r \leq 0$.

(d) Your plot should look like

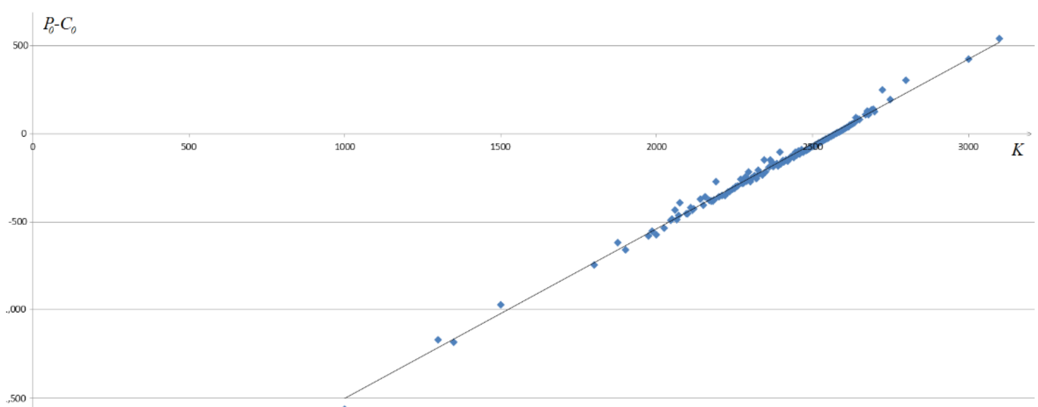

FIGURE 1. A plot of $P_0 - C_0$ versus K, where $t = 0$ corresponds to 23 October 2017 and and $t = T$ is 17 November 2017, and the underlying asset is the S&P 500 index with $S_0 = 2,573.82$. The price of the calls and puts is taken to be the last traded price on the day (as opposed to the bid or ask price). All data is taken from `https://uk.finance.yahoo.com`.

From $P_0 - C_0 = B_0K - S_0$, one can directly see that the intercept coefficient should be close to $-S_0$ and the coefficient for K should be close to B_0. For the statistical interpretation of what "close" means and how to analyze the output of a linear regression, as well as how to decide of the quality of the fitted model, you are encouraged to read Chapter 3 of "An Introduction to Statistical Learning" by R. Tibshirani et al.

The above plot was taken from lecture notes of the Advanced Financial Models taught at the University of Cambridge by Mike Tehranchi in 2017.

Solution 4.4

(a) An equivalent martingale measure solves $E_Q[S_1^1] = S_0^1$ (no discounting is needed since $r = 0$). By Lemma I.4.1, we can identify the measure Q with a vector $q = (q_1, q_2, q_3)^\top \in \mathbb{R}^3_{++}$ where $q_1 = Q(S_1^1 = 3)$, $q_1 = Q(S_1^1 = 2)$ and $q_1 = Q(S_1^1 = 1)$. Q being an EMM, give the following equations:

$$\begin{cases} 3q_1 + 2q_2 + q_3 = 2 \\ q_1 + q_2 + q_3 = 1 \\ q_1, q_2, q_2 > 0 \end{cases}$$

This gives that the set of all EMMs is given by

$$\mathbb{P} = \{(q_1, 1 - 2q_1, q_1) \quad | \quad 0 < q_1 < 1/2\}$$

(b) The terminal payoff of the call option is $(S_T - K)^+$.

(c) Denoting θ^0 and θ^1 the holding in the bond and the stock, to super-replicate the payout H, we are asked to minimize $\theta^0 + 2\theta^1$ subject to the constraints

$$\begin{cases} \theta^0 + 3\theta^1 \geq 1 \\ \theta^0 + 2\theta^1 \geq 0 \\ \theta^0 + \theta^1 \geq 0 \end{cases}$$

By Theorem I.7.2 and the bonus question of Problem 1, we now that there exist a vector $\theta^* = (\theta^{0*}, \theta^{1*})^\top$ such that

$$\theta^{0*} S_0^0 + \theta^{1*} S_0^1 = \sup_{Q \in \mathbb{P}} E_Q[H] = \sup_{0 < q_1 < 1/2} [q_1 + (1 - 2q_1) \cdot 0 + q_1 \cdot 0] = 1/2$$

Unfortunately, the Linear Programming duality principle only tells us how to calculate the seller's price but tells nothing about the strategy that would give that price. However, for this problem, we can easily see that $\theta^* = (-1/2, 1/2)^\top$.

(d) Similarly, the most expensive sub-replication cost is 0 and is attained for $\theta^* = (0, 0)^\top$.