Exercise 5.1 The goal of this exercise is to recall a few properties of stopping times and corresponding $\sigma$-algebras. Let $\tau$ be a stopping time w.r.t. a filtration $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$. Recall that

$$\mathcal{F}_\tau := \{ A \in \mathcal{F} : A \cap \{ \tau \leq k \} \in \mathcal{F}_k \text{ for all } k \in \mathbb{N}_0 \}.$$ 

(a) Show that $\mathcal{F}_\tau$ is a $\sigma$-algebra, and $\tau$ is $\mathcal{F}_\tau$-measurable.

(b) Suppose $\sigma, \tau$ are two stopping times with $\sigma \leq \tau$ a.s. Show that $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$. In particular, if $\tau \equiv k$ where $k \in \mathbb{N}_0$, $\mathcal{F}_\tau = \mathcal{F}_k$.

(c) Suppose $A \in \mathcal{F}$. Show that $\tau_A := \tau 1_A + \infty 1_{A^c}$ is a stopping time if and only if $A \in \mathcal{F}_\tau$.

(d) If $\tau, \sigma$ are two stopping times, then $\tau \vee \sigma$ and $\tau \wedge \sigma$ are stopping times, and $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$. Moreover, $\{ \sigma \leq \tau \} \in \mathcal{F}_{\tau \wedge \sigma}$ and $\{ \sigma = \tau \} \in \mathcal{F}_{\tau \wedge \sigma}$.

(e) A mapping $Y$ defined on $\{ \tau < \infty \}$ is $\mathcal{F}_\tau$-measurable if and only if for every $k \in \mathbb{N}_0$, $Y 1_{\{ \tau \leq k \}}$ is $\mathcal{F}_k$-measurable.

Solution 5.1

(a) Clearly $\Omega \cap \{ \tau \leq k \} = \{ \tau \leq k \} \in \mathcal{F}_k$ for all $k \in \mathbb{N}$, which shows $\Omega \in \mathcal{F}_\tau$. If $A \in \mathcal{F}_\tau$, then

$$A^c \cap \{ \tau \leq k \} = \{ \tau \leq k \} \setminus (A \cap \{ \tau \leq k \}) \in \mathcal{F}_k.$$

This shows $A^c \in \mathcal{F}_\tau$, so $\mathcal{F}_\tau$ is closed under the formation of complements. Now let $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_\tau$. Then

$$\left( \bigcup_{n \in \mathbb{N}} A_n \right) \cap \{ \tau \leq k \} = \bigcup_{n \in \mathbb{N}} (A_n \cap \{ \tau \leq k \}) \in \mathcal{F}_k, \forall k \in \mathbb{N}_0.$$

Therefore $\mathcal{F}_\tau$ is a $\sigma$-algebra.
Now we check \( \tau \) is \( \mathcal{F}_\tau \)-measurable. Note that \( \tau \) takes values in \( \mathbb{N}_0 \cup \{\infty\} \). So \( \{\tau \leq \infty\} = \Omega \in \mathcal{F}_\infty \) and we only need to check \( \{\tau \leq n\} \in \mathcal{F}_\tau \) for every \( n \in \mathbb{N}_0 \).

Let \( n \in \mathbb{N}_0 \) be fixed. For every \( k \in \mathbb{N}_0 \), we observe that
\[
\{\tau \leq n\} \cap \{\tau \leq k\} = \{\tau \leq n\} \subset \mathcal{F}_k \text{ if } n \leq k, \text{ and} \\
\{\tau \leq n\} \cap \{\tau \leq k\} = \{\tau \leq k\} \in \mathcal{F}_k \text{ if } n > k.
\]

Thus \( \tau \) is \( \mathcal{F}_\tau \)-measurable.

(b) Let \( A \in \mathcal{F}_\sigma \). The assumption \( \sigma \leq \tau \) implies \( \{\tau \leq k\} \subseteq \{\sigma \leq k\} \). Then for all \( k \in \mathbb{N}_0 \), we have
\[
A \cap \{\tau \leq k\} = (A \cap \{\sigma \leq k\}) \cap \{\tau \leq k\} \in \mathcal{F}_k
\]
because \( A \in \mathcal{F}_\sigma \). This shows \( A \in \mathcal{F}_\tau \) and \( \mathcal{F}_\sigma \subseteq \mathcal{F}_\tau \).

Now if \( \tau \equiv k \), then \( \mathcal{F}_\tau \subseteq \mathcal{F}_k \) and \( \mathcal{F}_k \subseteq \mathcal{F}_\tau \), which yields \( \mathcal{F}_\tau = \mathcal{F}_k \).

(c) Observe that for all \( k \in \mathbb{N}_0 \),
\[
\{\tau_A \leq k\} = \{\tau \leq k\} \cap A.
\]
This identity shows that \( \tau_A \) is a stopping time if and only if \( A \in \mathcal{F}_\tau \).

(d) The claim that \( \sigma \lor \tau, \sigma \land \tau \) are stopping times follow from the relations
\[
\{\sigma \lor \tau \leq k\} = \{\sigma \leq k\} \cap \{\tau \leq k\} \in \mathcal{F}_k
\]
and
\[
\{\sigma \land \tau \leq k\} = \{\sigma \leq k\} \cup \{\tau \leq k\} \in \mathcal{F}_k.
\]
Now because \( \sigma \land \tau \leq \sigma \) and \( \sigma \land \tau \leq \tau \), part (b) gives \( \mathcal{F}_{\sigma \land \tau} \subseteq \mathcal{F}_\sigma \cap \mathcal{F}_\tau \). Next suppose that \( A \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau \). We observe that
\[
A \cap \{\sigma \land \tau \leq k\} = A \cap (\{\sigma \leq k\} \cup \{\tau \leq k\})
\]
\[
= (A \cap \{\sigma \leq k\}) \cup (A \cap \{\tau \leq k\}) \in \mathcal{F}_k.
\]
This shows \( \mathcal{F}_\sigma \cap \mathcal{F}_\tau \subseteq \mathcal{F}_{\sigma \land \tau} \) and hence \( \mathcal{F}_{\sigma \land \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau \).

To prove the remaining claims, note that for each \( k \in \mathbb{N} \),
\[
\{\sigma \leq \tau\} \cap \{\tau \leq k\} = \bigcup_{i=0}^{k} (\{\sigma \leq \tau\} \cap \{\tau = i\}) = \bigcup_{i=0}^{k} (\{\sigma \leq i\} \cap \{\tau = i\}) \in \mathcal{F}_k.
\]
Thus \( \{\sigma \leq \tau\} \in \mathcal{F}_\tau \). Similarly, for each \( k \in \mathbb{N}_0 \), we have
\[
\{\sigma \leq \tau\} \cap \{\sigma \leq k\} = \{\sigma \land k \leq \tau \land k\} \cap \{\sigma \leq k\} \in \mathcal{F}_k
\]
because \( \sigma \land k, \tau \land k \) are both \( \mathcal{F}_k \)-measurable by parts (a) and (b). Hence \( \{\sigma \leq \tau\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \land \tau} \).

The very last assertion follows from \( \{\sigma = \tau\} = \{\sigma \leq \tau\} \cap \{\tau \leq \sigma\} \).
(e) The key identity is

\[ \{Y \leq a\} \cap \{\tau < \infty\} \cap \{\tau \leq k\} = \{Y \leq a\} \cap \{\tau \leq k\}, \forall k \in \mathbb{N}. \]

Hence \( Y \) on \( \{\tau < \infty\} \) is \( \mathcal{F}_\tau \)-measurable if and only if \( Y 1\{\tau \leq k\} \) is \( \mathcal{F}_k \)-measurable.
Exercise 5.2  Consider a financial market \((S^0, S^1)\) given by the following trees, where the numbers beside the branches denote transition probabilities.

\[
S^0 : 1 \quad 1 \quad 1 + r \quad 1 \quad (1 + r)(1 + r)
\]

\[
S^1 : 1 \quad \frac{1}{2} \quad 1 + u \quad \frac{1}{2} \quad (1 + u)(1 + 2u)
\]

\[
\frac{1}{2} \quad \frac{1}{2} \quad 1 + d \quad \frac{1}{2} \quad (1 + d)(1 + d)
\]

Intuitively, this means that the volatility of \(S^1\) increases if the stock price increases in the first period. Assume that \(u, r \geq 0\) and \(-0.5 < d \leq 0\).

(a) Construct for this setup a multiplicative model consisting of a probability space \((\Omega, \mathcal{F}, P)\), a filtration \(\mathcal{F} = (\mathcal{F}_k)_{k=0,1,2}\), two random variables \(Y_1\) and \(Y_2\) and two adapted stochastic processes \(S^0\) and \(S^1\) such that \(S^1_k = \prod_{j=1}^k Y_j\) for \(k = 0, 1, 2\).

(b) For which values of \(u\) and \(d\) are \(Y_1\) and \(Y_2\) uncorrelated?

(c) For which values of \(u\) and \(d\) are \(Y_1\) and \(Y_2\) independent?

(d) For which values of \(u\), \(r\) and \(d\) is the discounted stock process \(X^1 = S^1/S^0\) a \(P\)-martingale?

Solution 5.2

(a) We construct the canonical model for this setup, a path space. Let \(\Omega := \{-1, 1\}^2\), take \(\mathcal{F} := 2^\Omega\) and define \(P\) by

\[
P[\{(x_1, x_2)\}] := p_{x_1} p_{x_1, x_2},
\]

where \(p_{-1} := 1/2\) and \(p_{1,1} = p_{1, -1} = p_{-1,1} = p_{-1, -1} := 1/2\). Next, define \(Y_1\) and \(Y_2\) by

\[
Y_1((1, 1)) = Y_1((1, -1)) := 1 + u,
Y_1((-1, 1)) = Y_1((-1, -1)) := 1 + d, \text{ and }
Y_2((1, 1)) := 1 + 2u, Y_2((1, -1)) := 1 + 2d,
Y_2((-1, 1)) := 1 + u, Y_2((-1, -1)) := 1 + d.
\]
Finally, define \( S^0 \) and \( S^1 \) by \( S^0_k := (1 + r)^k \) and \( S^1_k := \prod_{j=1}^k Y_j \) for \( k = 0, 1, 2 \) and set \( \mathcal{F}_0 := \{\emptyset, \Omega\} \), \( \mathcal{F}_1 := \sigma(Y_1) = \{\emptyset, \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}, \Omega\} \) and \( \mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F} \).

(b) \( Y_1 \) and \( Y_2 \) are uncorrelated if and only if \( E[Y_1 Y_2] = E[Y_1] E[Y_2] \). Set \( c := (u + d)/2 \) to simplify the notation. Then we have

\[
E[Y_1] = 1 + c \quad \text{and} \quad E[Y_2] = 1 + \frac{3}{2} c,
\]

\[
E[Y_1 Y_2] = \frac{1 + u}{2}(1 + 2c) + \frac{1 + d}{2}(1 + c) = (1 + c)^2 + \frac{1 + u}{2} c.
\]

Hence, we have

\[
E[Y_1 Y_2] - E[Y_1] E[Y_2] = (1 + c)^2 + \frac{1 + u}{2} c - \left((1 + c)^2 + (1 + c)\frac{c}{2}\right)
\]

\[
= (u - c)\frac{c}{2}.
\]

Since \( d \leq 0 \leq u \), we have

\[
(u - c)\frac{c}{2} = 0 \iff c = 0 \quad \text{or} \quad u - c = 0 \iff d = -u.
\]

In conclusion, \( Y_1 \) and \( Y_2 \) are uncorrelated if and only if \( d = -u \).

(c) Since independence of two random variables implies that they are uncorrelated, we only have to consider the case in which \( u = -d \). If \( u = d = 0 \), \( Y_1 \) and \( Y_2 \) are both constant and hence independent. Otherwise, if \( u > 0 \), we have on the one hand

\[
P[Y_1 = 1 + u, Y_2 = 1 + u] = 0
\]

and on the other hand

\[
P[Y_1 = 1 + u] P[Y_2 = 1 + u] = 1/2 \cdot 1/4 = 1/8 \neq 0,
\]

showing that in this case \( Y_1 \) and \( Y_2 \) are not independent. In conclusion, \( Y_1 \) and \( Y_2 \) are independent if and only if \( u = d = 0 \).

Note: If \( d = -u \) and \( u \neq 0 \), then \( Y_1 \) and \( Y_2 \) are uncorrelated but not independent.

(d) \( X^1 \) is a \( P \)-martingale if and only if

\[
E\left[X_1^1 \bigg| \mathcal{F}_0\right] = X_0^1 \quad \text{P-a.s.} \quad \text{and} \quad E\left[X_2^1 \bigg| \mathcal{F}_1\right] = X_1^1 \quad \text{P-a.s.} \quad \text{(*)}
\]

If \( u = d = 0 \), it is straightforward to check that \( X^1 \) is a \( P \)-martingale if and only if \( r = 0 \). Next, assume that \( u > d \). Since \( \mathcal{F}_0 \) is trivial, \( \mathcal{F}_1 = \sigma(Y_1) \) and \( Y_1 > 0 \), \( (*) \) is equivalent to

\[
E[Y_1] = 1 + r \quad \text{and} \quad E[Y_2 \big| Y_1] = 1 + r \quad \text{P-a.s.}
\]
Since $Y_1$ only takes two values, this is equivalent to

\[ E[Y_1] = 1 + r \quad \text{and} \quad E[Y_2 | Y_1 = 1 + u] = 1 + r \quad \text{and} \quad E[Y_2 | Y_1 = 1 + d] = 1 + r. \]

This is equivalent to the linear system

\[
\begin{align*}
1 + (u + d)/2 &= 1 + r, \\
1 + u + d &= 1 + r, \\
1 + (u + d)/2 &= 1 + r.
\end{align*}
\]

Subtracting the first from the second equation yields $(u + d)/2 = 0$, which in turn implies $r = 0$. In conclusion, $X^1$ is a $P$-martingale if and only if $r = 0$ and $d = -u$. 
Exercise 5.3 Consider a market with trading dates $k = 0, \ldots, T$, with $N$ traded assets on the probability space $(\Omega, \mathcal{F}, P)$ and the filtration given by $\mathbb{F} = (\mathcal{F}_k)_{k=0,\ldots,T}$, i.e., a general multiperiod market.

For any strategy $\psi$, we define the process $\tilde{C} = (\tilde{C}_k)_{k=0,\ldots,T}$ by

$$\tilde{C}_k(\psi) := \tilde{V}_k(\psi) - \tilde{G}_k(\psi).$$

(a) Show that $\Delta \tilde{C}_{k+1}(\psi) = \Delta \psi_{k+1} \cdot S_k$ for $k = 1, \ldots, T - 1$.

(b) Show that $\psi$ is self-financing if and only if $\tilde{C}_k(\psi) = \tilde{C}_0(\psi)$ for $k = 0, \ldots, T$.

*Hint:* Be careful with the definitions at the first time point.

*Remark:* The process $\tilde{C}$ is called the *(undiscounted)* cost process for $\psi$.

(c) Suppose that $D = (D_k)_{k=0,\ldots,T}$ is an $\mathbb{R}$-valued strictly positive stochastic process adapted to $\mathbb{F}$. Define $Y_k = D_k S_k$ for $k = 0, \ldots, T$. Show that $\psi$ is self-financing for the price process $S = (S_k)_{k=0,1,\ldots,T}$ if and only if $\psi$ is self-financing for the price process $Y = (Y_k)_{k=0,1,\ldots,T}$.

Solution 5.3

(a) We need to show that $\Delta \tilde{V}_{k+1}(\psi) - \Delta \tilde{G}_{k+1}(\psi) = \Delta \psi_{k+1} \cdot S_k$ for $k = 1, \ldots, T - 1$. By the definitions,

$$\begin{align*}
\Delta \tilde{V}_{k+1}(\psi) - \Delta \tilde{G}_{k+1}(\psi) &= \psi_{k+1} \cdot S_{k+1} - \psi_k \cdot S_k - \psi_{k+1} \cdot \Delta S_{k+1} \\
&= -\psi_k \cdot S_k + \psi_{k+1} \cdot S_k \\
&= \Delta \psi_{k+1} \cdot S_k,
\end{align*}$$

which means we are done.

(b) The property $\tilde{C}_k(\psi) = \tilde{C}_0(\psi)$ for $k = 0, \ldots, T$ is equivalent to $\Delta \tilde{C}_{k+1} = 0$ for $k = 0, \ldots, T - 1$.

In view of (a), this condition looks stronger than $\psi$ being self-financing; so we need the observation that $\tilde{C}_1(\psi) = \tilde{C}_0(\psi)$ always holds. Indeed,

$$\tilde{C}_1(\psi) = \tilde{V}_1(\psi) - \tilde{G}_1(\psi) = \psi_1 \cdot S_1 - \psi_1 \cdot \Delta S_1 = \psi_1 \cdot S_0 = \tilde{V}_0(\psi) = \tilde{C}_0(\psi),$$

i.e., $\Delta \tilde{C}_1 = 0$ is always true. Combining this observation with (a), the definition of $\psi$ being self-financing is equivalent to $\Delta \tilde{C}_{k+1} = 0$ for $k = 0, \ldots, T - 1$. By the first equivalence, we are done.

\footnote{This shows that being self-financing is a numéraire-independent concept.}
(c) By definition, $\psi$ is self-financing if and only if for all $k \in \{0, 1, ..., T - 1\}$,

$$(\psi_{k+1} - \psi_k) \cdot S_k = 0.$$ 

Because $D$ is strictly positive, this is equivalent to

$$(\psi_{k+1} - \psi_k) \cdot S_k D_k = 0.$$ 

This means that $\psi$ is self-financing for $Y$. 
Exercise 5.4 Let \((S^0_t, S^1_t)\) be a model of an arbitrage-free complete financial market with two assets and a finite time horizon \(T\). Suppose that \(S^0\) is a numéraire asset satisfying \(S^0_{t+1} \geq S^0_t\) for all \(t \geq 0\). Let \(C(T, K)\) be the initial replication cost of a European Call option with strike \(K\) and maturity \(T\) written on the risky asset \(S^1\). The goal of this exercise is to show that \(T \rightarrow C(T, K)\) is increasing and that \(K \rightarrow C(T, K)\) is decreasing and convex.

(a) We define a martingale deflator to be an adapted process \(Y\) such that \(Y_t > 0\) for all \(t \geq 0\) almost surely and such that the process \(SY = (S_t Y_t)_{t \geq 0}\) is a martingale (under the original measure \(P\)). Show that there is a one-to-one correspondence between martingale deflectors and equivalent martingale measures (in finite time horizon models). Hint: Given a martingale deflator \(Y\), consider the measure \(Q\) defined by the Radon-Nykodym derivative
\[
\frac{dQ}{dP} = \frac{Y_T S^0_T}{E_P[Y_T S^0_T]}
\]
and show (using Bayes formula) that \(Q\) defined this way is indeed an EMM. Conversely, given an EMM \(Q\), consider the density process
\[
Z_t = E_P \left[ \frac{dQ}{dP} | \mathcal{F}_t \right]
\]
and show that the process \(Y\) defined by \(Y_t = \frac{Z_t}{S^0_t}\) is a martingale deflator.

Note that if \(Y\) is a martingale deflator, then so is \(cY\) for any \(c > 0\). In what follows we will consider the unique martingale deflator such that \(Y_0 = 1\).

(b) Let \(Y\) be the unique martingale deflator such that \(Y_0 = 1\). Show that \(Y\) is a \(P\)-supermartingale. Hint: for the integrability, you may use the fact that if the market model \(S\) with \(N\) assets is complete, then for each \(t \geq 0\) the probability space \(\Omega\) can be partitioned into no more than \(N^t\) \(\mathcal{F}_t\)-measurable events of positive probability. In particular, the \(N\)-dimensional random vector \(S_t\) takes values in a set of at most \(N^t\) elements and hence is bounded.

(c) Show that the process defined by \(Y_t(S^1_t - K)^+ = (Y_t S^1_t - Y_t K)^+\) is a \(P\)-submartingale.

(d) Write down the initial replication cost of a European Call option with strike \(K\) and maturity \(T\) as a function of the martingale deflator \(Y\).

(e) Conclude that \(T \rightarrow C(T, K)\) is increasing and that \(K \rightarrow C(T, K)\) is decreasing and convex.

(f) (Bonus) Using the programming language of your choice, verify the above monotonicity and convexity properties of the call surface on real historical data.
Solution 5.4

(a) Let $Y$ be a martingale deflator (note that in particular $Y_T S_T$ is $P$-integrable).

We want to construct an EMM $Q$. Therefore we define a new measure $Q$ with Radon-Nykodym density

$$\frac{dQ}{dP} = \frac{Y_T S_T^0}{E_P[Y_T S_T^0]}$$

Observe that

- $Q$ is a probability measure since $Q[\Omega] = E_Q[1_\Omega] = E_P\left[\frac{dQ}{dP}\right] = 1$
- $Q$ is equivalent to $P$ since $\frac{dQ}{dP} > 0$ by the positiveness of the martingale deflator $Y$.

Remains to show that $Q$ is a martingale measure. By Bayes formula,

$$E_Q\left[\frac{S^1_T}{S^0_T} | \mathcal{F}_t\right] = \frac{E_P[\frac{S^1_T Y_T}{S^0_T Y_T} | \mathcal{F}_t]}{E_P[\frac{S^0_T Y_T}{S^0_T Y_T} | \mathcal{F}_t]}$$

Since $Y$ is a martingale deflator, the numerator is equal to $E_P[\frac{S^1_T Y_T}{S^0_T Y_T} | \mathcal{F}_t] = S^1_t Y_t$ and the denominator is equal to $E_P[\frac{S^0_T Y_T}{S^0_T Y_T} | \mathcal{F}_t] = S^0_t Y_t$. Simplifying by the non-negative $Y_t$ gives

$$E_Q\left[\frac{S^1_T}{S^0_T} | \mathcal{F}_t\right] = \frac{S^1_t}{S^0_t}$$

and hence $Q$ is a martingale measure.

Conversely, suppose that $Q$ is an EMM. Let

$$Z_t = E_P\left[\frac{dQ}{dP} | \mathcal{F}_t\right]$$

Note that $Z$ is a $P$-martingale (prove it!). Moreover since $Q$ since equivalent to $P$, the process $Z$ is positive. Define

$$Y_t = \frac{Z_t}{S^0_t}$$

We now show that $Y$ is a martingale deflator. First, $Y$ is positive since $Z$ and $S^0$ are positive. Note that the process $Y$ satisfies

$$E_P\left[S^0_T Y_T | \mathcal{F}_t\right] = E_P\left[S^0_T Y_T | \mathcal{F}_t\right] = Z_t = S^0_t Y_t$$
Furthermore, \( S_1^t/S_T^0 \) is \( Q \)-integrable (by the definition of martingale) and hence \( S_1^T \) is \( P \)-integrable. We can thus conclude using Bayes formula that

\[
E_P \left[ S_1^T Y_T | \mathcal{F}_t \right] = E_Q \left[ \frac{S_1^T}{S_T^0} | \mathcal{F}_t \right] E_P \left[ S_T^0 Y_T | \mathcal{F}_t \right]
\]

\[
= \frac{S_1^t}{S_T^0} S_t^0 Y_t
\]

\[
= S_1^T Y_t
\]

so \( Y \) is a martingale deflator.

(b) Since the market is complete, there is no problem with integrability because \( Y_t \) is bounded for all \( t \geq 0 \). Using our assumption that \( S_{t+1}^0 \geq S_t^0 \) for all \( t \geq 0 \), we have

\[
Y_t \leq \frac{Y_t S_t^0}{S_s^0}
\]

and hence using that \( Y \) is a martingale deflator, we get

\[
E_P[Y_t | \mathcal{F}_s] \leq E_P \left[ \frac{Y_t S_t^0}{S_s^0} | \mathcal{F}_s \right]
\]

\[
= \frac{1}{S_s^0} E_P[Y_t S_t^0 | \mathcal{F}_s]
\]

\[
= \frac{Y_s S_s^0}{S_s^0} = Y_s
\]

Hence \( Y \) is a \( P \)-supermartingale.

(c) Jensen’s inequality and the martingale property of \( Y S_1^T \) together imply

\[
E_P[(Y_t S_t^1 - Y_t K)^+ | \mathcal{F}_s] \geq (E_P[Y_t S_t^1 - Y_t K | \mathcal{F}_s])^+
\]

\[
= (Y_s S_s^1 - K E_P[Y_t | \mathcal{F}_s])^+
\]

\[
\geq (Y_s S_s^1 - Y_s K)^+
\]

where the supermartingale property of \( Y \) has been used in the last line.

(d) By no arbitrage, we know form lecture that

\[
C(T, K) = E_Q \left[ (S_T^1 - K)^+ \right] / S_T^0
\]

Using the one-to-one correspondence between EMMs and martingale deflators given by

\[
\frac{dQ}{dP} = \frac{Y_T S_T^0}{E_P[Y_T S_T^0]}
\]

we conclude that

\[
C(T, K) = \frac{E_P[Y_T (S_T^1 - K)^+]}{S_T^0}
\]
(e) That $K \to C(T, K)$ is decreasing and convex is immediate from the same properties of $K \to (S^1_T - K)^+$. That $T \to C(T, K)$ is increasing is a consequence of the submartingale property of $Y(S^1 - K)^+$.

(f) We can plot the call surface in a 3D plot. A typical result should look like

![Call Surface Plot](image)

Figure 1: Figure taken from "Semi-nonparametric estimation of the call-option price surface under strike and time-to-expiry no-arbitrage constraints" by Mathias R. Fengler et al.