Introduction to Mathematical Finance

Exercise sheet 5

Exercise 5.1 The goal of this exercise is to recall a few properties of stopping times and corresponding σ-algebras. Let τ be a stopping time w.r.t. a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. Recall that

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \le k\} \in \mathcal{F}_k \text{ for all } k \in \mathbb{N}_0\}.$$

(a) Show that $\mathcal{F}_\tau$ is a σ-algebra, and τ is $\mathcal{F}_\tau$-measurable.

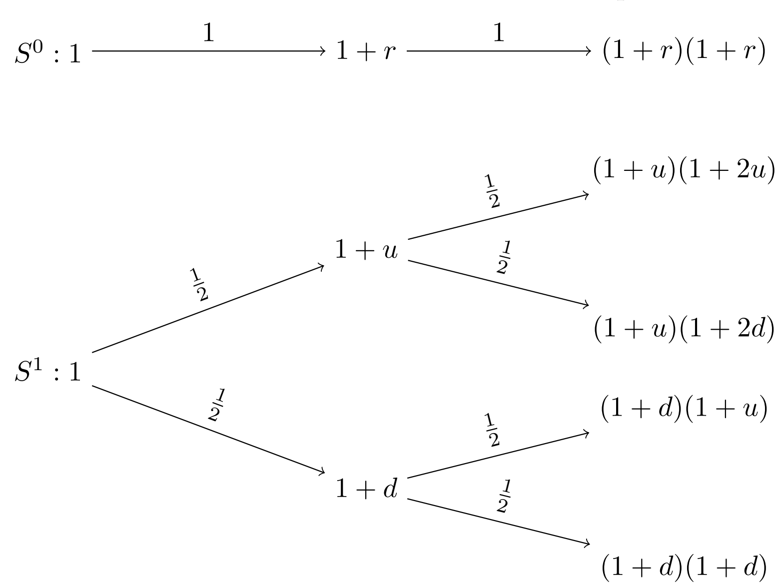
(b) Suppose σ, τ are two stopping times with $\sigma \le \tau$ P-a.s. Show that $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$. In particular, if $\tau \equiv k$ where $k \in \mathbb{N}_0$, $\mathcal{F}_\tau = \mathcal{F}_k$.

(c) Suppose $A \in \mathcal{F}$. Show that $\tau_A := \tau\mathbb{1}_A + \infty\mathbb{1}_{A^c}$ is a stopping time if and only if $A \in \mathcal{F}_\tau$.

(d) If τ, σ are two stopping times, then $\tau \vee \sigma$ and $\tau \wedge \sigma$ are stopping times, and $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau\wedge\sigma}$. Moreover, $\{\sigma \le \tau\} \in \mathcal{F}_{\tau\wedge\sigma}$ and $\{\sigma = \tau\} \in \mathcal{F}_{\tau\wedge\sigma}$.

(e) A mapping Y defined on $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable if and only if for every $k \in \mathbb{N}_0$, $Y\mathbb{1}\{\tau \le k\}$ is $\mathcal{F}_k$-measurable.

Exercise 5.2 Consider a financial market (S^0, S^1) given by the following trees, where the numbers beside the branches denote transition probabilities.



Intuitively, this means that the volatility of S^1 increases if the stock price increases in the first period. Assume that u, $r \geq 0$ and $-0.5 < d \leq 0$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes S^0 and S^1 such that $S^1_k = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$.

(b) For which values of u and d are Y_1 and Y_2 *uncorrelated*?

(c) For which values of u and d are Y_1 and Y_2 *independent*?

(d) For which values of u, r and d is the discounted stock process $X^1 = S^1/S^0$ a P-martingale?

Exercise 5.3 Consider a market with trading dates $k = 0, \ldots, T$, with N traded assets on the probability space $(\Omega, \mathcal{F}, P)$ and the filtration given by $\mathbb{F} = (\mathcal{F}_k)_{k=0,\ldots,T}$, i.e., a general multiperiod market.

For any strategy ψ, we define the process $\widetilde{C} = (\widetilde{C}_k)_{k=0,\ldots,T}$ by

$$\widetilde{C}_k(\psi) := \widetilde{V}_k(\psi) - \widetilde{G}_k(\psi).$$

(a) Show that

$$\Delta\widetilde{C}_{k+1}(\psi) = \Delta\psi_{k+1} \cdot S_k$$

for $k = 1, \ldots, T-1$.

(b) Show that ψ is self-financing if and only if

$$\widetilde{C}_k(\psi) = \widetilde{C}_0(\psi)$$

for $k = 0, \ldots, T$.

Hint: Be careful with the definitions at the first time point.

Remark: The process $\widetilde{C}$ is called the *(undiscounted) cost process* for ψ.

(c) Suppose that $D = (D_k)_{k=0,\ldots,T}$ is an $\mathbb{R}$-valued *strictly positive* stochastic process adapted to $\mathbb{F}$. Define $Y_k = D_k S_k$ for $k = 0, ..., T$. Show that ψ is self-financing for the price process $S = (S_k)_{k=0,1,\ldots,T}$ if and only if ψ is self-financing for the price process $Y = (Y_k)_{k=0,1,\ldots,T}$[1].

Exercise 5.4 Let (S^0_t, S^1_t) be a model of an arbitrage-free complete financial market with two assets and a finite time horizon T. Suppose that S^0 is a numéraire asset satisfying $S^0_{t+1} \geq S^0_t$ for all $t \geq 0$. Let $C(T, K)$ be the initial replication cost of a European Call option with strike K and maturity T written on the risky asset S^1. The goal of this exercise is to show that $T \to C(T, K)$ is increasing and that $K \to C(T, K)$ is decreasing and convex.

[1] This shows that being self-financing is a numéraire-independent concept.

(a) We define a *martingale deflator* to be an adapted process Y such that $Y_t > 0$ for all $t \geq 0$ almost surely and such that the process $SY = (S_t Y_t)_{t\geq 0}$ is a martingale (under the original measure P). Show that there is a one-to-one correspondence between martingale deflators and equivalent martingale measures (in finite time horizon models). *Hint: Given a martingale deflator Y, consider the measure Q defined by the Radon-Nykodym derivative*

$$\frac{dQ}{dP} = \frac{Y_T S_T^0}{E_P[Y_T S_T^0]}$$

and show (using Bayes formula) that Q defined this way is indeed an EMM. Conversely, given an EMM Q, consider the density process

$$Z_t = E_P\left[\frac{dQ}{dP}|\mathcal{F}_t\right]$$

and show that the process Y defined by $Y_t = \frac{Z_t}{S_t^0}$ is a martingale deflator.

Note that if Y is a martingale deflator, then so is cY for any $c > 0$. In what follows we will consider the unique martingale deflator such that $Y_0 = 1$.

(b) Let Y be the unique martingale deflator such that $Y_0 = 1$. Show that Y is a P-supermartingale. *Hint: for the integrability, you may use the fact that if the market model S with N assets is complete, then for each $t \geq 0$ the probability space Ω can be partitioned into no more than N^t $\mathcal{F}_t$-measurable events of positive probability. In particular, the N-dimensional random vector S_t takes values in a set of at most N^t elements and hence is bounded.*

(c) Show that the process defined by $Y_t(S_t^1 - K)^+ = (Y_t S_t^1 - Y_t K)^+$ is a P-submartingale.

(d) Write down the initial replication cost of a European Call option with strike K and maturity T as a function of the martingale deflator Y.

(e) Conclude that $T \to C(T, K)$ is increasing and that $K \to C(T, K)$ is decreasing and convex.

(f) *(Bonus)* Using the programming language of your choice, verify the above monotonicity and convexity properties of the call surface on real historical data.