

Introduction to Mathematical Finance

Exercise sheet 5

Exercise 5.1 The goal of this exercise is to recall a few properties of stopping times and corresponding \( \sigma \)-algebras. Let \( \tau \) be a stopping time w.r.t. a filtration \( \mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0} \). Recall that

\[
\mathcal{F}_\tau := \{ A \in \mathcal{F} : A \cap \{ \tau \leq k \} \in \mathcal{F}_k \text{ for all } k \in \mathbb{N}_0 \}.
\]

(a) Show that \( \mathcal{F}_\tau \) is a \( \sigma \)-algebra, and \( \tau \) is \( \mathcal{F}_\tau \)-measurable.

(b) Suppose \( \sigma, \tau \) are two stopping times with \( \sigma \leq \tau \) \( \mathbb{P} \)-a.s. Show that \( \mathcal{F}_\sigma \subset \mathcal{F}_\tau \). In particular, if \( \tau \equiv k \) where \( k \in \mathbb{N}_0 \), \( \mathcal{F}_\tau = \mathcal{F}_k \).

(c) Suppose \( A \in \mathcal{F} \). Show that \( \tau_A := \tau 1_A + \infty 1_{A^c} \) is a stopping time if and only if \( A \in \mathcal{F}_\tau \).

(d) If \( \tau, \sigma \) are two stopping times, then \( \tau \lor \sigma \) and \( \tau \land \sigma \) are stopping times, and \( \mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \land \sigma} \). Moreover, \( \{ \sigma \leq \tau \} \in \mathcal{F}_{\tau \land \sigma} \) and \( \{ \sigma = \tau \} \in \mathcal{F}_{\tau \land \sigma} \).

(e) A mapping \( Y \) defined on \( \{ \tau < \infty \} \) is \( \mathcal{F}_\tau \)-measurable if and only if for every \( k \in \mathbb{N}_0 \), \( Y 1_{\{ \tau \leq k \}} \) is \( \mathcal{F}_k \)-measurable.

Exercise 5.2 Consider a financial market \( (S^0, S^1) \) given by the following trees, where the numbers beside the branches denote transition probabilities.

\[
S^0 : 1 \xrightarrow{1} 1 + r \xrightarrow{1} (1 + r)(1 + r)
\]

\[
S^1 : 1 \xrightarrow{\frac{1}{2}} 1 + u \xrightarrow{\frac{1}{2}} (1 + u)(1 + 2u)
\]

\[
\xrightarrow{\frac{1}{2}} 1 + d \xrightarrow{\frac{1}{2}} (1 + d)(1 + d)
\]

\[
\xrightarrow{\frac{1}{2}} (1 + u)(1 + 2d)
\]

\[
(1 + d)(1 + u)
\]

\[
(1 + d)(1 + d)
\]
Intuitively, this means that the volatility of $S^1$ increases if the stock price increases in the first period. Assume that $u, r \geq 0$ and $-0.5 < d \leq 0$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables $Y_1$ and $Y_2$ and two adapted stochastic processes $S^0$ and $S^1$ such that $S^1_k = \prod_{j=1}^{k} Y_j$ for $k = 0, 1, 2$.

(b) For which values of $u$ and $d$ are $Y_1$ and $Y_2$ uncorrelated?

(c) For which values of $u$ and $d$ are $Y_1$ and $Y_2$ independent?

(d) For which values of $u$, $r$ and $d$ is the discounted stock process $X^1 = S^1 / S^0$ a $P$-martingale?

**Exercise 5.3** Consider a market with trading dates $k = 0, \ldots, T$, with $N$ traded assets on the probability space $(\Omega, \mathcal{F}, P)$ and the filtration given by $\mathbb{F} = (\mathcal{F}_k)_{k=0,\ldots,T}$, i.e., a general multiperiod market.

For any strategy $\psi$, we define the process $\tilde{C} = (\tilde{C}_k)_{k=0,\ldots,T}$ by

$$\tilde{C}_k(\psi) := \tilde{V}_k(\psi) - \tilde{G}_k(\psi).$$

(a) Show that

$$\Delta \tilde{C}_{k+1}(\psi) = \Delta \psi_{k+1} \cdot S_k$$

for $k = 1, \ldots, T - 1$.

(b) Show that $\psi$ is self-financing if and only if

$$\tilde{C}_k(\psi) = \tilde{C}_0(\psi)$$

for $k = 0, \ldots, T$.

**Hint:** Be careful with the definitions at the first time point.

**Remark:** The process $\tilde{C}$ is called the (undiscounted) cost process for $\psi$.

(c) Suppose that $D = (D_k)_{k=0,\ldots,T}$ is an $\mathbb{R}$-valued strictly positive stochastic process adapted to $\mathbb{F}$. Define $Y_k = D_k S_k$ for $k = 0, \ldots, T$. Show that $\psi$ is self-financing for the price process $S = (S_k)_{k=0,1,\ldots,T}$ if and only if $\psi$ is self-financing for the price process $Y = (Y_k)_{k=0,1,\ldots,T}$.

**Exercise 5.4** Let $(S^0_t, S^1_t)$ be a model of an arbitrage-free complete financial market with two assets and a finite time horizon $T$. Suppose that $S^0$ is a numéraire asset satisfying $S^0_{t+1} \geq S^0_t$ for all $t \geq 0$. Let $C(T, K)$ be the initial replication cost of a European Call option with strike $K$ and maturity $T$ written on the risky asset $S^1$. The goal of this exercise is to show that $T \to C(T, K)$ is increasing and that $K \to C(T, K)$ is decreasing and convex.

\footnote{This shows that being self-financing is a numéraire-independent concept.}

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(a) We define a martingale deflator to be an adapted process $Y$ such that $Y_t > 0$ for all $t \geq 0$ almost surely and such that the process $SY = (S_t Y_t)_{t \geq 0}$ is a martingale (under the original measure $P$). Show that there is a one-to-one correspondence between martingale deflators and equivalent martingale measures (in finite time horizon models). Hint: Given a martingale deflator $Y$, consider the measure $Q$ defined by the Radon-Nykodym derivative

$$\frac{dQ}{dP} = \frac{Y_T S_0^T}{E_P[Y_T S_0^T]}$$

and show (using Bayes formula) that $Q$ defined this way is indeed an EMM. Conversely, given an EMM $Q$, consider the density process

$$Z_t = E_P \left[ \frac{dQ}{dP} | F_t \right]$$

and show that the process $Y$ defined by $Y_t = \frac{Z_t}{S_t}$ is a martingale deflator.

Note that if $Y$ is a martingale deflator, then so is $cY$ for any $c > 0$. In what follows we will consider the unique martingale deflator such that $Y_0 = 1$.

(b) Let $Y$ be the unique martingale deflator such that $Y_0 = 1$. Show that $Y$ is a $P$-supermartingale. Hint: for the integrability, you may use the fact that if the market model $S$ with $N$ assets is complete, then for each $t \geq 0$ the probability space $\Omega$ can be partitioned into no more than $N^t$ $F_t$-measurable events of positive probability. In particular, the $N$-dimensional random vector $S_t$ takes values in a set of at most $N^t$ elements and hence is bounded.

(c) Show that the process defined by $Y_t(S_t^1 - K)^+ = (Y_t S_t^1 - Y_t^t K)^+$ is a $P$-submartingale.

(d) Write down the initial replication cost of a European Call option with strike $K$ and maturity $T$ as a function of the martingale deflator $Y$.

(e) Conclude that $T \to C(T, K)$ is increasing and that $K \to C(T, K)$ is decreasing and convex.

(f) (Bonus) Using the programming language of your choice, verify the above monotonicity and convexity properties of the call surface on real historical data.