Introduction to Mathematical Finance

Solution sheet 5

Solution 5.1

(a) Clearly $\Omega \cap \{\tau \leq k\} = \{\tau \leq k\} \in \mathcal{F}_k$ for all $k \in \mathbb{N}$, which shows $\Omega \in \mathcal{F}_\tau$. If $A \in \mathcal{F}_\tau$, then

$$A^c \cap \{\tau \leq k\} = \underbrace{\{\tau \leq k\}}_{\in \mathcal{F}_k} \setminus \underbrace{(A \cap \{\tau \leq k\})}_{\in \mathcal{F}_k} \in \mathcal{F}_k.$$

This shows $A^c \in \mathcal{F}_\tau$, so $\mathcal{F}_\tau$ is closed under the formation of complements. Now let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}_\tau$. Then

$$\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \cap \{\tau \leq k\} = \bigcup_{n\in\mathbb{N}} \underbrace{(A_n \cap \{\tau \leq k\})}_{\in \mathcal{F}_k} \in \mathcal{F}_k,\ \forall k \in \mathbb{N}_0.$$

Therefore $\mathcal{F}_\tau$ is a σ-algebra.

Now we check τ is $\mathcal{F}_\tau$-measurable. Note that τ takes values in $\mathbb{N}_0 \cup \{\infty\}$. So $\{\tau \leq \infty\} = \Omega \in \mathcal{F}_\infty$ and we only need to check $\{\tau \leq n\} \in \mathcal{F}_\tau$ for every $n \in \mathbb{N}_0$. Let $n \in \mathbb{N}_0$ be fixed. For every $k \in \mathbb{N}_0$, we observe that

$$\begin{aligned}
&\{\tau \leq n\} \cap \{\tau \leq k\} = \{\tau \leq n\} \in \mathcal{F}_n \subset \mathcal{F}_k \text{ if } n \leq k, \text{ and}\\
&\{\tau \leq n\} \cap \{\tau \leq k\} = \{\tau \leq k\} \in \mathcal{F}_k \text{ if } n > k.
\end{aligned}$$

Thus τ is $\mathcal{F}_\tau$-measurable.

(b) Let $A \in \mathcal{F}_\sigma$. The assumption $\sigma \leq \tau$ implies $\{\tau \leq k\} \subseteq \{\sigma \leq k\}$. Then for all $k \in \mathbb{N}_0$, we have

$$A \cap \{\tau \leq k\} = (\underbrace{A \cap \{\sigma \leq k\}}_{\in \mathcal{F}_k}) \cap \{\tau \leq k\} \in \mathcal{F}_k$$

because $A \in \mathcal{F}_\sigma$. This shows $A \in \mathcal{F}_\tau$ and $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

Now if $\tau \equiv k$, then $\mathcal{F}_\tau \subseteq \mathcal{F}_k$ and $\mathcal{F}_k \subseteq \mathcal{F}_\tau$, which yields $\mathcal{F}_\tau = \mathcal{F}_k$.

(c) Observe that for all $k \in \mathbb{N}_0$,

$$\{\tau_A \leq k\} = \{\tau \leq k\} \cap A.$$

This identity shows that τ_A is a stopping time if and only if $A \in \mathcal{F}_\tau$.

(d) The claim that $\sigma \vee \tau, \sigma \wedge \tau$ are stopping times follow from the relations

$$\{\sigma \vee \tau \leq k\} = \{\sigma \leq k\} \cap \{\tau \leq k\} \in \mathcal{F}_k$$

and

$$\{\sigma \wedge \tau \leq k\} = \{\sigma \leq k\} \cup \{\tau \leq k\} \in \mathcal{F}_k.$$

Now because $\sigma \wedge \tau \leq \sigma$ and $\sigma \wedge \tau \leq \tau$, part (b) gives $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. Next suppose that $A \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. We observe that

$$\begin{aligned} A \cap \{\sigma \wedge \tau \leq k\} &= A \cap (\{\sigma \leq k\} \cup \{\tau \leq k\}) \\ &= (\underbrace{A \cap \{\sigma \leq k\}}_{\in \mathcal{F}_k}) \cup (\underbrace{A \cap \{\tau \leq k\}}_{\in \mathcal{F}_k}) \in \mathcal{F}_k. \end{aligned}$$

This shows $\mathcal{F}_\sigma \cap \mathcal{F}_\tau \subseteq \mathcal{F}_{\sigma \wedge \tau}$ and hence $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

To prove the remaining claims, note that for each $k \in \mathbb{N}$,

$$\{\sigma \leq \tau\} \cap \{\tau \leq k\} = \bigcup_{i=0}^{k} (\{\sigma \leq \tau\} \cap \{\tau = i\}) = \bigcup_{i=0}^{k} (\{\sigma \leq i\} \cap \{\tau = i\}) \in \mathcal{F}_k.$$

Thus $\{\sigma \leq \tau\} \in \mathcal{F}_\tau$. Similarly, for each $k \in \mathbb{N}_0$, we have

$$\{\sigma \leq \tau\} \cap \{\sigma \leq k\} = \{\sigma \wedge k \leq \tau \wedge k\} \cap \{\sigma \leq k\} \in \mathcal{F}_k$$

because $\sigma \wedge k, \tau \wedge k$ are both $\mathcal{F}_k$-measurable by parts (a) and (b). Hence $\{\sigma \leq \tau\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \wedge \tau}$.

The very last assertion follows from $\{\sigma = \tau\} = \{\sigma \leq \tau\} \cap \{\tau \leq \sigma\}$.

(e) The key identity is

$$\{Y \leq a\} \cap \{\tau < \infty\} \cap \{\tau \leq k\} = \{Y \leq a\} \cap \{\tau \leq k\}, \forall k \in \mathbb{N}.$$

Hence Y on $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable if and only if $Y \mathbb{1}\{\tau \leq k\}$ is $\mathcal{F}_k$-measurable.

Solution 5.2

(a) We construct the canonical model for this setup, a path space. Let $\Omega := \{-1, 1\}^2$, take $\mathcal{F} := 2^\Omega$ and define P by

$$P\left[\{(x_1, x_2)\}\right] := p_{x_1} p_{x_1, x_2},$$

where $p_1 = p_{-1} := 1/2$ and $p_{1,1} = p_{1,-1} = p_{-1,1} = p_{-1,-1} := 1/2$. Next, define Y_1 and Y_2 by

$$\begin{aligned} &Y_1((1,1)) = Y_1((1,-1)) := 1 + u, \\ &Y_1((-1,1)) = Y_1((-1,-1)) := 1 + d, \text{ and} \\ &Y_2((1,1)) := 1 + 2u, Y_2((1,-1)) := 1 + 2d, \\ &Y_2((-1,1)) := 1 + u, Y_2((-1,-1)) := 1 + d. \end{aligned}$$

Finally, define S^0 and S^1 by $S_k^0 := (1+r)^k$ and $S_k^1 := \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$ and set $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1) = \{\emptyset, \{(1,1),(1,-1)\}, \{(-1,1),(-1,-1)\}, \Omega\}$ and $\mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F}$.

(b) Y_1 and Y_2 are uncorrelated if and only if $E[Y_1 Y_2] = E[Y_1]\,E[Y_2]$. Set $c := (u+d)/2$ to simplify the notation. Then we have

$$E[Y_1] = 1 + c \quad \text{and} \quad E[Y_2] = 1 + \frac{3}{2}c\,,$$
$$E[Y_1 Y_2] = \frac{1+u}{2}(1+2c) + \frac{1+d}{2}(1+c) = (1+c)^2 + \frac{1+u}{2}c\,.$$

Hence, we have

$$\begin{aligned} E[Y_1 Y_2] - E[Y_1]\,E[Y_2] &= (1+c)^2 + \frac{1+u}{2}c - \Big((1+c)^2 + (1+c)\frac{c}{2}\Big) \\ &= (u-c)\frac{c}{2}\,. \end{aligned}$$

Since $d \le 0 \le u$, we have

$$(u-c)\frac{c}{2} = 0 \quad \Longleftrightarrow \quad c = 0 \text{ or } u - c = 0 \quad \Longleftrightarrow \quad d = -u\,.$$

In conclusion, Y_1 and Y_2 are uncorrelated if and only if $d = -u$.

(c) Since independence of two random variables implies that they are uncorrelated, we only have to consider the case in which $u = -d$. If $u = d = 0$, Y_1 and Y_2 are both constant and hence independent. Otherwise, if $u > 0$, we have on the one hand

$$P[Y_1 = 1+u, Y_2 = 1+u] = 0$$

and on the other hand

$$P[Y_1 = 1+u]\,P[Y_2 = 1+u] = 1/2 \cdot 1/4 = 1/8 \neq 0\,,$$

showing that in this case Y_1 and Y_2 are not independent. In conclusion, Y_1 and Y_2 are independent if and only if $u = d = 0$.

Note: If $d = -u$ and $u \neq 0$, then Y_1 and Y_2 are uncorrelated but **not** independent.

(d) X^1 is a P-martingale if and only if

$$E\Big[X_1^1 \,\Big|\, \mathcal{F}_0\Big] = X_0^1 \quad P\text{-a.s.} \quad \text{and} \quad E\Big[X_2^1 \,\Big|\, \mathcal{F}_1\Big] = X_1^1 \quad P\text{-a.s.} \qquad (*)$$

If $u = d = 0$, it is straightforward to check that X^1 is a P-martingale if and only if $r = 0$. Next, assume that $u > d$. Since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and $Y_1 > 0$, $(*)$ is equivalent to

$$E[Y_1] = 1 + r \quad \text{and} \quad E[Y_2 \,|\, Y_1] = 1 + r \quad P\text{-a.s.}$$

Since Y_1 only takes two values, this is equivalent to

$$E[Y_1] = 1+r \quad \text{and} \quad E[Y_2 \,|\, Y_1 = 1+u] = 1+r \quad \text{and} \quad E[Y_2 \,|\, Y_1 = 1+d] = 1+r\,.$$

This is equivalent to the linear system

$$\begin{aligned} 1 + (u+d)/2 &= 1 + r\,, \\ 1 + u + d &= 1 + r\,, \\ 1 + (u+d)/2 &= 1 + r\,. \end{aligned}$$

Subtracting the first from the second equation yields $(u+d)/2 = 0$, which in turn implies $r = 0$. In conclusion, X^1 is a P-martingale if and only if $r = 0$ and $d = -u$.

Solution 5.3

(a) We need to show that $\Delta\widetilde{V}_{k+1}(\psi) - \Delta\widetilde{G}_{k+1}(\psi) = \Delta\psi_{k+1} \cdot S_k$ for $k = 1, \dots, T-1$. By the definitions,

$$\begin{aligned} \Delta\widetilde{V}_{k+1}(\psi) - \Delta\widetilde{G}_{k+1}(\psi) &= \psi_{k+1} \cdot S_{k+1} - \psi_k \cdot S_k - \psi_{k+1} \cdot \Delta S_{k+1} \\ &= -\psi_k \cdot S_k + \psi_{k+1} \cdot S_k \\ &= \Delta\psi_{k+1} \cdot S_k, \end{aligned}$$

which means we are done.

(b) The property $\widetilde{C}_k(\psi) = \widetilde{C}_0(\psi)$ for $k = 0, \dots, T$ is equivalent to

$$\Delta\widetilde{C}_{k+1} = 0,$$

for $k = 0, \dots, T-1$.

In view of (a), this condition looks stronger than ψ being self-financing; so we need the observation that $\widetilde{C}_1(\psi) = \widetilde{C}_0(\psi)$ always holds. Indeed,

$$\widetilde{C}_1(\psi) = \widetilde{V}_1(\psi) - \widetilde{G}_1(\psi) = \psi_1 \cdot S_1 - \psi_1 \cdot \Delta S_1 = \psi_1 \cdot S_0 = \widetilde{V}_0(\psi) = \widetilde{C}_0(\psi),$$

i.e., $\Delta\widetilde{C}_1 = 0$ is always true. Combining this observation with (a), the definition of ψ being self-financing is equivalent to $\Delta\widetilde{C}_{k+1} = 0$ for $k = 0, \dots, T-1$. By the first equivalence, we are done.

(c) By definition, ψ is self-financing if and only if for all $k \in \{0, 1, ..., T-1\}$,

$$(\psi_{k+1} - \psi_k) \cdot S_k = 0.$$

Because D is strictly positive, this is equivalent to

$$(\psi_{k+1} - \psi_k) \cdot S_k D_k = 0.$$

This means that ψ is self-financing for Y.

Solution 5.4

(a) Let Y be a martingale deflator (note that in particular $Y_T S_T$ is P-integrable). We want to construct an EMM Q. Therefore we define a new measure Q with Radon-Nykodym density

$$\frac{dQ}{dP} = \frac{Y_T S_T^0}{E_P[Y_T S_T^0]}$$

Observe that

- Q is a probability measure since $Q[\Omega] = E_Q[\mathbb{1}_\Omega] = E_P\left[\frac{dQ}{dP}\right] = 1$
- Q is equivalent to P since $\frac{dQ}{dP} > 0$ by the positiveness of the martingale deflator Y.

Remains to show that Q is a martingale measure. By Bayes formula,

$$E_Q\left[\frac{S_T^1}{S_T^0}|\mathcal{F}_t\right] = \frac{E_P[S_T^1 Y_T|\mathcal{F}_t]}{E_P[S_T^0 Y_T|\mathcal{F}_t]}$$

Since Y is a martingale deflator, the numerator is equal to $E_P[S_T^1 Y_T|\mathcal{F}_t] = S_t^1 Y_t$ and the denominator is equal to $E_P[S_T^0 Y_T|\mathcal{F}_t] = S_t^0 Y_t$. Simplifying by the non-negative Y_t gives

$$E_Q\left[\frac{S_T^1}{S_T^0}|\mathcal{F}_t\right] = \frac{S_t^1}{S_t^0}$$

and hence Q is a martingale measure.

Conversely, suppose that Q is an EMM. Let

$$Z_t = E_P\left[\frac{dQ}{dP}|\mathcal{F}_t\right]$$

Note that Z is a P-martingale (prove it!). Moreover since Q since equivalent to P, the process Z is positive. Define

$$Y_t = \frac{Z_t}{S_t^0}$$

We now show that Y is a martingale deflator. First, Y is positive since Z and S^0 are positive. Note that the process Y satisfies

$$\begin{aligned} E_P\left[S_T^0 Y_T|\mathcal{F}_t\right] &= E_P\left[S_T^0 Y_T|\mathcal{F}_t\right] \\ &= Z_t \\ &= S_t^0 Y_t \end{aligned}$$

Furthermore, S^1_T/S^0_T is Q-integrable (by the definition of martingale) and hence $S^1_T Y_T$ is P-integrable. We can thus conclude using Bayes formula that

$$\begin{aligned} E_P\left[S^1_T Y_T|\mathcal{F}_t\right] &= E_Q\left[\frac{S^1_T}{S^0_T}|\mathcal{F}_t\right] E_P\left[S^0_T Y_T|\mathcal{F}_t\right] \\ &= \frac{S^1_t}{S^0_t} S^0_t Y_t \\ &= S^1_t Y_t \end{aligned}$$

so Y is a martingale deflator.

(b) Since the market is complete, there is no problem with integrability because Y_t is bounded for all $t \geq 0$. Using our assumption that $S^0_{t+1} \geq S^0_t$ for all $t \geq 0$, we have

$$Y_t \leq \frac{Y_t S^0_t}{S^0_s}$$

and hence using that Y is a martingale deflator, we get

$$\begin{aligned} E_P[Y_t|\mathcal{F}_s] &\leq E_P\left[\frac{Y_t S^0_t}{S^0_s}|\mathcal{F}_s\right] \\ &= \frac{1}{S^0_s} E_P[Y_t S^0_t|\mathcal{F}_s] \\ &= \frac{Y_s S^0_s}{S^0_s} = Y_s \end{aligned}$$

Hence Y is a P-supermartingale.

(c) Jensen's inequality and the martingale property of YS^1 together imply

$$\begin{aligned} E_P[(Y_t S^1_t - Y_t K)^+|\mathcal{F}_s] &\geq (E_P[Y_t S^1_t - Y_t K|\mathcal{F}_s])^+ \\ &= (Y_s S^1_s - K E_P[Y_t|\mathcal{F}_s])^+ \\ &\geq (Y_s S^1_s - Y_s K)^+ \end{aligned}$$

where the supermartingale property of Y has been used in the last line.

(d) By no arbitrage, we know form lecture that

$$C(T,K) = E_Q\left[\frac{(S^1_T - K)^+}{S^0_T}\right]$$

Using the one-to-one correspondence between EMMs and martingale deflators given by

$$\frac{dQ}{dP} = \frac{Y_T S^0_T}{E_P[Y_T S^0_T]}$$

we conclude that

$$C(T,K) = \frac{E_P\left[Y_T(S^1_T - K)^+\right]}{S^0_0}$$

(e) That $K \to C(T,K)$ is decreasing and convex is immediate from the same properties of $K \to (S^1_T - K)^+$. That $T \to C(T,K)$ is increasing is a consequence of the submartingale property of $Y(S^1 - K)^+$.

(f) We can plot the call surface in a 3D plot. A typical result should look like

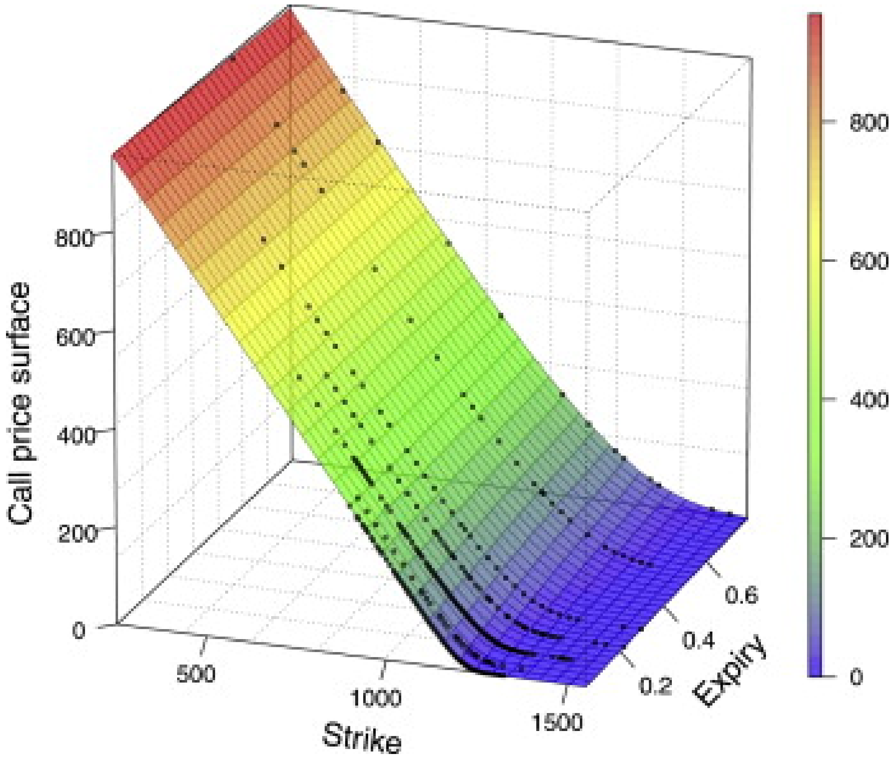

Figure 1: Figure taken from "Semi-nonparametric estimation of the call-option price surface under strike and time-to-expiry no-arbitrage constraints" by Mathias R. Fengler et al.