Introduction to Mathematical Finance

Exercise sheet 6

Exercise 6.1 The goal of this exercise is to study empirical properties of financial data. Moreover, we also illustrate the stylized facts that GARCH models can replicate and discuss the limitations of these models. Work through the *stylized_facts.R*-file.

Solution 6.1 Solutions are included in the R-file.

Exercise 6.2 Suppose that S^0 and S^1 are both numéraires. Denote by X and Y the discounted price processes w.r.t. S^0 and S^1, respectively, and $\mathbb{P}(X), \mathbb{P}(Y)$ the corresponding sets of EMMs.

(a) Show that $\mathbb{P}(X) \neq \varnothing \iff \mathbb{P}(Y) \neq \varnothing$.

(b) Show that

$$\mathbb{P}(Y) = \left\{ \tilde{Q} : \frac{d\tilde{Q}}{dQ} = \frac{X_T^1}{X_0^1} \text{ for some } Q \in \mathbb{P}(X) \right\}.$$

(c) Show that if X_T^1 is not P-a.s. constant, then $\mathbb{P}(X) \cap \mathbb{P}(Y) = \varnothing$.

Solution 6.2

(a) Since the notion of absence of arbitrage is invariant under any change of numéraire (see Exercise 5.3 (c)), we see that

$$\mathbb{P}(X) \neq \varnothing \iff \text{Both markets are arbitrage-free} \iff \mathbb{P}(Y) \neq \varnothing.$$

(b) Denote the collection on the RHS by $\mathcal{C}$. First we observe the following relation bewteen X and Y:

$$Y_k = X_k \frac{S_k^0}{S_k^1} = \frac{X_k}{X_k^1}, \quad \forall k = 0, 1, ..., T.$$

Suppose that $Q \in \mathbb{P}(X)$. Then $(X_k^1/X_0^1)_{k=0}^T$ is clearly a positive martingale under Q with Q-expectation 1. So $D := d\tilde{Q}/dQ = X_T^1/X_0^1$ induces a probability measure. Moreover, under $\tilde{Q}$, we compute for $j \leq k$ by using the Bayes rule that

$$\begin{aligned} E_{\tilde{Q}}\left[Y_k \,|\, \mathcal{F}_j\right] &= \frac{E_Q\left[Y_k E_Q\left[D \,|\, \mathcal{F}_k\right] \,|\, \mathcal{F}_j\right]}{E_Q\left[D \,|\, \mathcal{F}_j\right]} \\ &= \frac{E_Q\left[Y_k X_k^1/X_0^1 \,|\, \mathcal{F}_j\right]}{X_j^1/X_0^1} \\ &= \frac{E_Q\left[X_k \,|\, \mathcal{F}_j\right]}{X_j^1} \\ &= \frac{X_j}{X_j^1} = Y_j. \end{aligned}$$

Thus Y is a martingale under $\tilde{Q}$ and $\mathcal{C} \subseteq \mathbb{P}(Y)$. Reversing the roles of X and Y yields

$$\left\{ Q : \frac{dQ}{d\tilde{Q}} = \frac{Y_T^0}{Y_0^0} \text{ for some } \tilde{Q} \in \mathbb{P}(Y) \right\} \subset \mathbb{P}(X).$$

Finally, observe $Y^0_T/Y^0_0 = (X^1_T/X^1_0)^{-1}$. Therefore, for any $\tilde{Q} \in \mathbb{P}(Y)$, we have

$$d\tilde{Q} = \frac{X^1_T}{X^1_0} \underbrace{\frac{Y^0_T}{Y^0_0} d\tilde{Q}}_{\in \mathbb{P}(X)}.$$

This shows $\mathbb{P}(Y) \subset \mathcal{C}$ and the equality follows.

(c) Since the function $x \mapsto 1/x$ is strictly convex on $(0, \infty)$, we obtain

$$E_{\tilde{Q}}[Y^0_T] = Y^0_0 = \frac{1}{X^1_0} = \frac{1}{E_Q[X^1_T]} < E_Q\left[\frac{1}{X^1_T}\right] = E_Q[Y^0_T], \quad \forall Q \in \mathbb{P}(X), \tilde{Q} \in \mathbb{P}(Y).$$

So if there is $Q_0 \in \mathbb{P}(X) \cap \mathbb{P}(Y)$, then we obtain from above that $E_{Q_0}[Y^0_T] < E_{Q_0}[Y^0_T]$, which is a contradiction.

Exercise 6.3 Consider a market with only 1 risky asset. Let $X_k = (X_k^1)_{k=0,...,T}$ be the P-a.s. strictly positive discounted price process. Recall that the returns are defined by

$$R_k := \frac{X_k - X_{k-1}}{X_{k-1}}, \quad k = 1, ..., T,$$

so that

$$X_k = X_0 \prod_{i=1}^{k} (1 + R_i).$$

We take the filtration $\mathcal{F}_k = \sigma(X_0, ..., X_k)$.

(a) Show that X is a martingale if $(R_k)_{k=1,...,T}$ are independent and integrable random variables with $E[R_k] = 0$.

(b) Now give necessary and sufficient conditions on $(R_k)_{k=1,...,T}$ such that X is a martingale.

(c) Construct an example in which X is a martingale but the returns $(R_k)_{k=1,...,T}$ are not independent.

Solution 6.3

(a) Suppose the R_k are independent with mean 0. Clearly X is adapted to its own filtration. Also using the independence of R_k and $E[R_k] = 0$, we have

$$E[X_k] = X_0 \prod_{i=1}^{k} E[1 + R_i] = X_0 < \infty.$$

We now check the martingale property:

$$E\left[X_k \,|\, \mathcal{F}_{k-1}\right] = X_{k-1} E[1 + R_k] = X_{k-1}, \quad k = 1, 2, ..., T.$$

Thus X is a martingale.

(b) The process X is clearly adapted to its natural filtration. So we only need to impose conditions on (R_k) so that X is integrable and X satisfies the martingale property.

Claim. X is a martingale if and only if $R_k > -1$ P-a.s., $E\left[R_k \,|\, \mathcal{F}_{k-1}\right] = 0$ and $E[|R_k|] \leq 2$ for $k = 1, \ldots, T$.

Proof of Claim. "$\Longrightarrow$" Suppose that X is a martingale. Then $E\left[\frac{X_k}{X_{k-1}} \,\middle|\, \mathcal{F}_{k-1}\right] = 1$ and $\frac{X_k}{X_{k-1}} = 1 + R_k \geq 0$; so $R_k \geq -1$, $R_k^- \leq 1 \in L^1$ and $E[1 + R_k] = 1$ or $E[R_k] = 0$. But then $0 = E[R_k] = E[R_k^+ - R_k^-] = E[R_k^+] - E[R_k^-]$ so that $E[R_k^+] = E[R_k^-] \leq 1$ and $E[|R_k^+|] \leq 2$.

"$\Longleftarrow$" Suppose that the converse is true. Then $E[X_k] \leq X_0^1(1+2)^T < \infty$ for $k = 1, ..., T$. So X is integrable. Then we compute

$$E\left[\frac{X_k}{X_{k-1}}\middle|\mathcal{F}_{k-1}\right] = E\left[1 + R_k \,|\, \mathcal{F}_{k-1}\right] = 1,$$

which shows $E\left[X_k \,|\, \mathcal{F}_{k-1}\right] = X_{k-1}$ for $k = 1, ..., T$. Therefore, X is a martingale.

(c) Note that in Exercise 5.2, we constructed such a multiplicative model. There, the returns are independent iff $u = d = 0$. But the discounted price process is a P-martingale as long as $r = 0$ and $u = -d$.

Exercise 6.4 Let ψ given by (V_0, ϑ) be a self-financing strategy in a multiperiod market with discounted asset prices $(1, X) = S/S^0$. Assume that $V_T(\psi) \geq -a$ P-a.s. for some $a \geq 0$.

(a) Show that if the market is arbitrage-free, then ψ is a-admissible, i.e., $V_k(\psi) \geq -a$ P-a.s. for all $k = 0, \dots, T$.

(b) Show, without using (a), that if X admits an ELMM Q and $V_0 \in L^1(Q)$, then $V_k(\psi) \geq -a$ P-a.s. for all $k = 0, \dots, T$.

Solution 6.4

(a) Suppose that ψ is not a-admissible. Then there exists a time point k for which the event $A = \{V_k(\psi) < -a\}$ has strictly positive probability, i.e., $P[A] > 0$. Let k_0 be the largest such time point. Construct the self-financing strategy $\psi' = (\psi'^0, \vartheta')$ described by $V_0 = 0$ and

$$\vartheta'_k = \begin{cases} 0 & \text{if } k \neq k_0 + 1, \\ \vartheta_k \mathbb{1}_A & \text{if } k = k_0 + 1. \end{cases}$$

Note that this process is indeed predictable since ϑ is predictable and $A \in \mathcal{F}_{k_0}$, and that ψ' is well defined by Proposition II.1.2. We first compute, using that $\vartheta \triangle X = \triangle G(\vartheta) = \triangle V(\psi)$ that

$$V_k(\psi') = V_0 + G_k(\vartheta') = 0 + \mathbb{1}_{\{k \geq k_0+1\}} \mathbb{1}_A \vartheta_k \triangle X_k = \mathbb{1}_{\{k \geq k_0+1\}} \mathbb{1}_A \triangle V_{k_0+1}(\psi).$$

By definition of A and k_0, this is nonnegative P-a.s., and strictly positive on A, hence with positive probability. So ψ' is 0-admissible and $V_0(\psi') = 0$, $V_T(\psi') \in L^0_+ \setminus \{0\}$ so that ψ' is an arbitrage opportunity.

(b) By assumption, X is a local Q-martingale. Therefore, by Proposition C.4, $G(\vartheta)$ (hence $V(\psi)$) is a local Q-martingale. Furthermore,

$$E_Q[|V_0(\psi)|] < \infty$$

and $E_Q[V_T^-(\psi)] \leq a$, so from Theorem C.5 we conclude that $V(\psi)$ is a (true) Q-martingale.

By the martingale property,

$$V_k(\psi) = E[V_T(\psi) | \mathcal{F}_k] \geq -a \quad Q\text{-a.s.},$$

for all $t = 0, \dots, T$, thus also P-a.s., which is what we wanted to show.

Exercise 6.5 Consider a discrete time model with N **dividend paying** assets. Let δ_t^i be the dividend payment at time t per share of asset i, and let S_t^i be the ex-dividend price of the asset at time t, i.e. the price of the asset immediately after the dividend is paid.

(a) Explain why an appropriate self-financing condition for a strategy $\psi \in \mathbb{R}^N$ (without consumption/pure investment) is

$$\psi_t \cdot (S_t + \delta_t) = \psi_{t+1} \cdot S_t$$

(b) Suppose that there exists a positive process Z such that the process

$$M_t = Z_t S_t + \sum_{s=1}^{t} Z_s \delta_s$$

defines a martingale. Show that under this assumption, the market is arbitrage-free.

Hint: You may use the following proposition. Let M be a martingale and K a predictable process (in discrete time) and let

$$N_t = N_0 + \sum_{s=1}^{t} K_s (M_s - M_{s-1})$$

Then $(N_t)_{0 \le t \le T}$ is a local martingale. Suppose moreover that there exists a non-random time $T > 0$ such that $N_T \ge 0$. Then $(N_t)_{0 \le t \le T}$ is a true martingale.

(c) Suppose that the dividend process δ is non-negative. Show that there exists a self-financing (pure investment) trading strategy with corresponding wealth process

$$\tilde{V}_t = \tilde{V}_t(\psi) = S_t \prod_{s=1}^{t} \left(1 + \frac{\delta_s}{S_s}\right)$$

Give a financial interpretation of your strategy.

Solution 6.5

(a) At the end of the t-th trading period, the investor's wealth in a given asset is the sum of the ex-dividend price of the asset and the dividend payout. The investor's total wealth at time t is thus $\psi_t \cdot (S_t + \delta_t)$. Under self-financing assumption, the investor uses this money to rebalance his portfolio for the next period so that

$$\psi_t \cdot (S_t + \delta_t) = \psi_{t+1} \cdot S_t$$

as required.

(b) Let

$$M_t = Z_t S_t + \sum_{s=1}^{t} Z_s \delta_s$$

and consider the consumption stream

$$c_0 = -\psi_1 \cdot S_0$$
$$c_t = 0 \quad \forall 0 < t < T \quad \text{(no consumption between t=0 and t=T)}$$
$$c_T = \psi_T \cdot S_T$$

Define

$$N_t = Z_t \psi_{t+1} \cdot S_t + \sum_{s=0}^{t} Z_s c_s$$

By calculating the increments $N_t - N_{t-1}$ and $M_t - M_{t-1}$, one can easily see that N can be written as a discrete time stochastic integral of ψ with respect to M:

$$N_t = \sum_{s=1}^{t} \psi_s (M_s - M_{s-1})$$

Assuming that M is a martingale implies that N is a local martingale. Take an arbitrage candidate, i.e. a strategy ψ such that $c_0 \geq 0$ and $c_T \geq 0$. We need to show that ψ cannot be an arbitrage, i.e. that we must have $c_0 = c_T = 0$. For such an arbitrage candidate ψ, note that $N_T = c_0 Z_0 + c_T Z_t \geq 0$ (since the process Z is positive). Hence N is a true martingale by the hint. Therefore $E[N_T] = N_0 = 0$. By the pigeonhole principle, $N_T = 0$ almost surely, and hence $c_t = 0 \quad \forall 0 \leq t \leq T$ since Z is supposed to be a positive process. In particular ψ cannot be an arbitrage and hence the market is arbitrage-free.

(c) Let

$$\psi_t = \frac{\tilde{V}_{t-1}}{S_{t-1}}$$

Notice that ψ is clearly predictable. This portfolio consists of holding buying one share of the asset at time $t = 0$ and then reinvesting the dividend payments at all time. In particular no external funding is required and hence we expect this strategy to be self-financing. More precisely, the strategy is self-financing (without consumption) since

$$\psi_t (S_t + \delta_t) = \tilde{V}_t = \psi_{t+1} S_t$$