Introduction to Mathematical Finance

Exercise sheet 6

Exercise 6.1 The goal of this exercise is to study empirical properties of financial data. Moreover, we also illustrate the stylized facts that GARCH models can replicate and discuss the limitations of these models. Work through the *stylized_facts.R*-file.

Exercise 6.2 Suppose that S^0 and S^1 are both numéraires. Denote by X and Y the discounted price processes w.r.t. S^0 and S^1, respectively, and $\mathbb{P}(X), \mathbb{P}(Y)$ the corresponding sets of EMMs.

(a) Show that $\mathbb{P}(X) \neq \varnothing \iff \mathbb{P}(Y) \neq \varnothing$.

(b) Show that

$$\mathbb{P}(Y) = \left\{ \tilde{Q} : \frac{d\tilde{Q}}{dQ} = \frac{X_T^1}{X_0^1} \text{ for some } Q \in \mathbb{P}(X) \right\}.$$

(c) Show that if X_T^1 is not P-a.s. constant, then $\mathbb{P}(X) \cap \mathbb{P}(Y) = \varnothing$.

Exercise 6.3 Consider a market with only 1 risky asset. Let $X_k = (X_k^1)_{k=0,...,T}$ be the P-a.s. strictly positive discounted price process. Recall that the returns are defined by

$$R_k := \frac{X_k - X_{k-1}}{X_{k-1}}, \quad k = 1, ..., T,$$

so that

$$X_k = X_0 \prod_{i=1}^{k} (1 + R_i).$$

We take the filtration $\mathcal{F}_k = \sigma(X_0, ..., X_k)$.

(a) Show that X is a martingale if $(R_k)_{k=1,...,T}$ are independent and integrable random variables with $E[R_k] = 0$.

(b) Now give necessary and sufficient conditions on $(R_k)_{k=1,...,T}$ such that X is a martingale.

(c) Construct an example in which X is a martingale but the returns $(R_k)_{k=1,...,T}$ are not independent.

Exercise 6.4 Let ψ given by (V_0, ϑ) be a self-financing strategy in a multiperiod market with discounted asset prices $(1, X) = S/S^0$. Assume that $V_T(\psi) \geq -a$ P-a.s. for some $a \geq 0$.

(a) Show that if the market is arbitrage-free, then ψ is a-admissible, i.e., $V_k(\psi) \geq -a$ P-a.s. for all $k = 0, \dots, T$.

(b) Show, without using (a), that if X admits an ELMM Q and $V_0 \in L^1(Q)$, then $V_k(\psi) \geq -a$ P-a.s. for all $k = 0, \dots, T$.

Exercise 6.5 Consider a discrete time model with N **dividend paying** assets. Let δ_t^i be the dividend payment at time t per share of asset i, and let S_t^i be the ex-dividend price of the asset at time t, i.e. the price of the asset immediately after the dividend is paid.

(a) Explain why an appropriate self-financing condition for a strategy $\psi \in \mathbb{R}^N$ (without consumption/pure investment) is

$$\psi_t \cdot (S_t + \delta_t) = \psi_{t+1} \cdot S_t$$

(b) Suppose that there exists a positive process Z such that the process

$$M_t = Z_t S_t + \sum_{s=1}^{t} Z_s \delta_s$$

defines a martingale. Show that under this assumption, the market is arbitrage-free.

Hint: You may use the following proposition. Let M be a martingale and K a predictable process (in discrete time) and let

$$N_t = N_0 + \sum_{s=1}^{t} K_s (M_s - M_{s-1})$$

Then $(N_t)_{0 \leq t \leq T}$ is a local martingale. Suppose moreover that there exists a non-random time $T > 0$ such that $N_T \geq 0$. Then $(N_t)_{0 \leq t \leq T}$ is a true martingale.

(c) Suppose that the dividend process δ is non-negative. Show that there exists a self-financing (pure investment) trading strategy with corresponding wealth process

$$\tilde{V}_t = \tilde{V}_t(\psi) = S_t \prod_{s=1}^{t} \left(1 + \frac{\delta_s}{S_s}\right)$$

Give a financial interpretation of your strategy.